Slides for April 23

Math 24 — Spring 2014

Rank of a Matrix

Definition

If $A \in \mathrm{M}_{m \times n}(F)$ then the **rank of** A is the rank of the left multiplication transformation $L_A : F^n \to F^m$.

Theorem

If $x_1, x_2, \ldots, x_n$ *are the columns of the* $m \times n$ *matrix* A, *then*

$$\mathrm{rank}(A) = \dim(\mathrm{span}\{x_1, x_2, \ldots, x_n\}).$$

Proof.

Because $\mathrm{R}(L_A) = \mathrm{span}\{L_A(e_1), L_A(e_2), \ldots, L_A(e_n)\}$ by Theorem 2.2 and

$$x_1 = Ae_1 = L_A(e_1), x_2 = Ae_2 = L_A(e_2), \ldots, x_n = Ae_n = L_A(e_n).$$

□

Rank and Bases

Theorem

If α *and* β *are ordered bases for* V *and* W, *respectively, and* $T : \mathsf{V} \to \mathsf{W}$ *is a linear transformation then* $\text{rank}(T) = \text{rank}([T]_\alpha^\beta)$.

Corollary

If P *is an invertible* $m \times m$ *matrix,* Q *is an invertible* $n \times n$ *matrix,* A *is an* $m \times n$ *matrix, then* $\text{rank}(PAQ^{-1}) = \text{rank}(A)$.

Proof.

Let $\alpha = \{x_1, \ldots, x_n\}$ be the columns of Q, $\beta = \{y_1, \ldots, y_m\}$ be the columns of P.
Then $P = [I_{F^m}]_\beta^{\mathrm{std}}$ and $Q^{-1} = [I_{F^n}]_{\mathrm{std}}^\alpha$. Therefore $PAQ^{-1} = [L_A]_\alpha^\beta$ and so

$$\text{rank}(A) = \text{rank}(L_A) = \text{rank}([L_A]_\alpha^\beta) = \text{rank}(PAQ^{-1}).$$

□

Computing Rank

Theorem 3.6

Any $m \times n$ matrix A can be put into the form

$$\begin{pmatrix} I_r & O_{r\times(n-r)} \\ O_{(m-r)\times r} & O_{(m-r)\times(n-r)} \end{pmatrix}$$

using a sequence of elementary row and column operations.

Therefore there are elementary matrices $E_1, \ldots, E_p$ and $C_1, \ldots, C_q$ such that

$$\begin{pmatrix} I_r & O \\ O & O \end{pmatrix} = (E_1 \cdots E_p)A(C_1 \cdots C_q).$$

Since elementary matrices are invertible, we have

$$r = \operatorname{rank}\left(\begin{pmatrix} I_r & O \\ O & O \end{pmatrix}\right) = \operatorname{rank}(A).$$

Echelon Form

Definition

A matrix is in **echelon form** if the first nonzero entry in any row is the only nonzero entry in the rectangle extending from there to the lower left corner of the matrix. The first nonzero entry in a row is called the **pivot** of that row.

$$\begin{pmatrix} \circledast & * & * & * & * & * & \cdots \\ 0 & 0 & \circledast & * & * & * & \cdots \\ 0 & 0 & 0 & \circledast & * & * & \cdots \\ 0 & 0 & 0 & 0 & 0 & \circledast & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

Echelon Form

Theorem

Any matrix can be put into echelon form by a sequence of elementary row operations.

Theorem

The rank of a matrix in echelon form is the number of pivots of the matrix.

Because using only elementary column operations, all the pivots can be rescaled to 1's, moved to the left, and used to clear all other nonzero entries in their row. This process results in a matrix of the form:

$$\begin{pmatrix} 1 & \cdots & 0 & 0 & \cdots \\ \vdots & \ddots & \vdots & \vdots & \\ 0 & \cdots & 1 & 0 & \cdots \\ 0 & \cdots & 0 & 0 & \cdots \\ \vdots & & \vdots & \vdots & \ddots \end{pmatrix}.$$