Slides for April 9

Math 24 — Spring 2014

Dimension Theorem

- The **nullity** of a linear transformation $T : \mathrm{V} \to \mathrm{W}$ is the dimension of the null space $\mathrm{N}(T) = \{v \in \mathrm{V} : T(v) = 0\}$.
- The **rank** of a linear transformation $T : \mathrm{V} \to \mathrm{W}$ is the dimension of the range space $\mathrm{R}(T) = \{T(v) \in \mathrm{W} : v \in \mathrm{V}\}$.

Dimension Theorem

If $T : \mathrm{V} \to \mathrm{W}$ is a linear transformation and V is finite dimensional, then

$$\mathrm{nullity}(T) + \mathrm{rank}(T) = \dim(\mathrm{V}).$$

The Dimension Theorem is also known as the Rank–Nullity Theorem.

Dimension Theorem

Start with a basis $\{v_1, \ldots, v_k\}$ of $\mathrm{N}(T)$ and extend it to a basis $\{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ for all of V.

Claim

$\{T(v_{k+1}), \ldots, T(v_n)\}$ is a basis for $\mathrm{R}(T)$.

Proof of Claim

We know that $\{T(v_1), \ldots, T(v_n)\}$ generates $\mathrm{R}(T)$.
Since $T(v_1) = T(v_2) = \cdots = T(v_k) = 0$, the subset $\{T(v_{k+1}), \ldots, T(v_n)\}$ already generates $\mathrm{R}(T)$.
It remains to see that $\{T(v_{k+1}), \ldots, T(v_n)\}$ is linearly independent...

Dimension Theorem

Start with a basis $\{v_1, \dots, v_k\}$ of $\mathrm{N}(T)$ and extend it to a basis $\{v_1, \dots, v_k, v_{k+1}, \dots, v_n\}$ for all of V.

Claim

$\{T(v_{k+1}), \dots, T(v_n)\}$ is a basis for $\mathrm{R}(T)$.

Suppose, $a_{k+1}, \dots, a_n$ are scalars such that

$$a_{k+1}T(v_{k+1}) + \cdots + a_n T(v_n) = 0.$$

Because T is linear, we see that

$$T(a_{k+1}v_{k+1} + \cdots + a_n v_n) = 0.$$

Therefore $a_{k+1}v_{k+1} + \cdots + a_n v_n$ is in the null space of T.

Dimension Theorem

Start with a basis $\{v_1, \ldots, v_k\}$ of $\mathrm{N}(T)$ and extend it to a basis $\{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ for all of V.

Claim

$\{T(v_{k+1}), \ldots, T(v_n)\}$ is a basis for $\mathrm{R}(T)$.

Since $\{v_1, \ldots, v_k\}$ is a basis for $\mathrm{N}(T)$, there are scalars $b_1, \ldots, b_k$ such that

$$a_{k+1}v_{k+1} + \cdots + a_n v_n = b_1 v_1 + \cdots + b_k v_k,$$

or equivalently

$$-b_1 v_1 - \cdots - b_k v_k + a_{k+1}v_{k+1} + \cdots + a_n v_n = 0.$$

Since $\{v_1, \ldots, v_n\}$ is linearly independent, we conclude that

$$-b_1 = \cdots = -b_k = a_{k+1} = \cdots = a_n = 0.$$

Dimension Theorem

Start with a basis $\{v_1, \ldots, v_k\}$ of N(T) and extend it to a basis $\{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ for all of V.

Claim

$\{T(v_{k+1}), \ldots, T(v_n)\}$ is a basis for R(T).

It follows that

$$\mathrm{rank}(T) = n - k = \dim(V) - \mathrm{nullity}(T),$$

or equivalently that

$$\mathrm{nullity}(T) + \mathrm{rank}(T) = \dim(\mathrm{V}).$$

One-to-One Linear Transformations

A linear transformation is **one-to-one** if

$$T(x) = T(y) \quad \text{implies} \quad x = y.$$

Theorem

A linear transformation $T : \mathrm{V} \to \mathrm{W}$ *is one-to-one if and only if* $\mathrm{N}(T) = \{0\}$.

($\Rightarrow$) If $T : \mathrm{V} \to \mathrm{W}$ is one-to-one, then $\mathrm{N}(T)$ can only have one element, which must be the zero vector.

One-to-One Linear Transformations

A linear transformation is **one-to-one** if

$$T(x) = T(y) \quad \text{implies} \quad x = y.$$

Theorem

A linear transformation $T : \mathrm{V} \to \mathrm{W}$ *is one-to-one if and only if* $\mathrm{N}(T) = \{0\}$.

($\Leftarrow$) Suppose $\mathrm{N}(T) = \{0\}$. If $T(x) = T(y)$, then

$$T(x - y) = T(x) - T(y) = 0.$$

Therefore $x - y \in \mathrm{N}(T)$. Since $\mathrm{N}(T) = \{0\}$ this means $x - y = 0$, or $x = y$.

One-to-One Linear Transformations

A linear transformation is **one-to-one** if

$$T(x) = T(y) \quad \text{implies} \quad x = y.$$

Theorem

A linear transformation $T : \mathsf{V} \to \mathsf{W}$ *is one-to-one if and only if* $\mathsf{N}(T) = \{0\}$.

Corollary

Suppose V *and* W *are finite dimensional vector spaces. For any linear transformation* $T : \mathsf{V} \to \mathsf{W}$, *the following are equivalent:*

1. T *is one-to-one*
2. $\text{nullity}(T) = 0$
3. $\text{rank}(T) = \dim(\mathsf{V})$

Onto Linear Transformations

A linear transformation $T : \mathrm{V} \to \mathrm{W}$ is **onto** if for every $w \in \mathrm{W}$ there is a $v \in \mathrm{V}$ such that $w = T(v)$.

Theorem

A linear transformation $T : \mathrm{V} \to \mathrm{W}$ is onto if and only if $\mathrm{R}(T) = \mathrm{W}$.

Corollary

Suppose V and W are finite dimensional vector spaces. For any linear transformation $T : \mathrm{V} \to \mathrm{W}$, the following are equivalent:

1. *T is onto*
2. $\mathrm{rank}(T) = \dim(\mathrm{W})$
3. $\mathrm{nullity}(T) = \dim(\mathrm{V}) - \dim(\mathrm{W})$

One-to-One and Onto Linear Transformations

Theorem

Suppose V *and* W *are finite dimensional vector spaces of equal dimension* n*. For any linear transformation* $T : \mathrm{V} \to \mathrm{W}$*, the following are equivalent:*

1. T *is one-to-one and onto*
2. T *is one-to-one*
3. T *is onto*
4. $\mathrm{nullity}(T) = 0$
5. $\mathrm{rank}(T) = n$