Slides for April 4

Math 24 — Spring 2014

Theorem 1

If $v_1, v_2, \ldots, v_k$ is a finite list of vectors in a vector space V such that

$$v_i \notin \text{span}\{v_1, \ldots, v_{i-1}\}$$

for $i = 1, 2, \ldots, k$, then the set $\{v_1, v_2, \ldots, v_k\}$ is linearly independent.

Indirect Proof.

Suppose

$$a_1 v_1 + a_2 v_2 + \cdots + a_i v_i = 0$$

where $a_i \neq 0$.
Then

$$v_i = -\frac{a_1}{a_i} v_1 - \frac{a_2}{a_i} v_2 - \cdots - \frac{a_{i-1}}{a_i} v_{i-1}.$$

Therefore, $v_i \in \text{span}\{v_1, v_2, \ldots, v_{i-1}\}$.

□

Theorem 2

Suppose A is a finite set of vectors in a vector space V. If $C \subseteq A$ is linearly independent then there is a linearly independent set B such that $C \subseteq B \subseteq A$ and $\text{span}(B) = \text{span}(A)$.

Write $A = \{v_1, v_2, \ldots, v_k\}$ and $C = \{v_1, v_2, \ldots, v_i\}$. We proceed by induction on $k \geq i$.

Base Case ($k = i$).

Then $A = C$ and $B = A = C$ works:

1. $C \subseteq B \subseteq A$.
2. B is linearly independent because $B = C$.
3. $\text{span}(B) = \text{span}(A)$ because $B = A$.

Theorem 2

Suppose A is a finite set of vectors in a vector space V. If $C \subseteq A$ is linearly independent then there is a linearly independent set B such that $C \subseteq B \subseteq A$ and $\text{span}(B) = \text{span}(A)$.

Induction Step ($k \to k+1$).

Let $A = \{v_1, v_2, \ldots, v_k, v_{k+1}\}$ be given.
Write $A_0 = \{v_1, v_2, \ldots, v_k\}$.
By the *Induction Hypothesis*, there is a set B_0 such that

1. $C \subseteq B_0 \subseteq A_0$.
2. B_0 is linearly independent.
3. $\text{span}(B_0) = \text{span}(A_0)$.

Theorem 2

Suppose A is a finite set of vectors in a vector space V. If $C \subseteq A$ is linearly independent then there is a linearly independent set B such that $C \subseteq B \subseteq A$ and $\text{span}(B) = \text{span}(A)$.

Induction Step (continued)

- If $v_{k+1} \in \text{span}(B_0)$ then $B = B_0$ works:
  1. $C \subseteq B \subseteq A$.
  2. B is linearly independent because $B = B_0$.
  3. $\text{span}(B) = \text{span}(A)$ because $A = A_0 \cup \{v_{k+1}\} \subseteq \text{span}(B)$.
- If $v_{k+1} \notin \text{span}(B_0)$ then $B = B_0 \cup \{v_{k+1}\}$ works:
  1. $C \subseteq B \subseteq A$.
  2. B is linearly independent by Theorem 1.7.
  3. $\text{span}(B) = \text{span}(A)$ because $A = A_0 \cup \{v_{k+1}\} \subseteq \text{span}(B)$.

Theorem 3

Every finite generating set in a vector space V contains a basis for V.

Let A be a finite generating subset of V.
By Theorem 2, there is a set B such that:

1. $\varnothing \subseteq B \subseteq A$.
2. B is linearly independent.
3. $\text{span}(B) = \text{span}(A) = \text{V}$.

Thus, B is a basis for V contained in A.

Theorem 4

Every finite linearly independent set in a finitely generated vector space V can be extended to a basis for V.

Proof.

Let C be a finite linearly independent set and let A be a finite generating for V set containing C.
By Theorem 2, there is a set B such that:

1. $C \subseteq B \subseteq A$.
2. B is linearly independent.
3. $\text{span}(B) = \text{span}(A) = \text{V}$.

Thus, B is a basis for V extending C. $\square$

Theorem 5

If $v_1, v_2, \ldots, v_k$ is a finite list of vectors in a vector space V then every list of $k+1$ (or more) vectors from $\text{span}\{v_1, v_2, \ldots, v_k\}$ is linearly dependent.

We proceed by induction on $k \geq 1$.

Base Case ($k = 1$).

Suppose $x_1, x_2 \in \text{span}\{v_1\}$, say $x_1 = a_1 v_1$ and $x_2 = a_2 v_1$.
Then

$$a_2 x_1 - a_1 x_2 = a_2(a_1 v_1) - a_1(a_2 v_1) = (a_2 a_1 - a_1 a_2) v_1 = 0 v_1 = 0.$$

So, if $a_1 \neq 0$ or $a_2 \neq 0$, this shows that x_1, x_2 are linearly dependent.
On the other hand, if $a_1 = a_2 = 0$ then $x_1 = x_2 = 0$ and hence x_1, x_2 are again linearly dependent.

Theorem 5

If $v_1, v_2, \ldots, v_k$ is a finite list of vectors in a vector space V then every list of $k+1$ (or more) vectors from span$\{v_1, v_2, \ldots, v_k\}$ is linearly dependent.

Induction Step ($k-1 \to k$).

Suppose $x_1, x_2, \ldots, x_k, x_{k+1} \in \text{span}\{v_1, v_2, \ldots, v_k\}$, say:

$$\begin{array}{rcccccc}
x_1 & = & a_{1,1}v_1 & + & a_{1,2}v_2 & + \cdots + & a_{1,k}v_k \\
x_2 & = & a_{2,1}v_1 & + & a_{2,2}v_2 & + \cdots + & a_{2,k}v_k \\
\vdots & & \vdots & & & & \vdots \\
x_k & = & a_{k,1}v_1 & + & a_{k,2}v_2 & + \cdots + & a_{k,k}v_k \\
x_{k+1} & = & a_{k+1,1}v_1 & + & a_{k+1,2}v_2 & + \cdots + & a_{k+1,k}v_k
\end{array}$$

Theorem 5

If $v_1, v_2, \ldots, v_k$ is a finite list of vectors in a vector space V then every list of $k+1$ (or more) vectors from $\text{span}\{v_1, v_2, \ldots, v_k\}$ is linearly dependent.

Induction Step (continued).

- In the case where $a_{1,k} = a_{2,k} = \cdots = a_{k,k} = a_{k+1,k} = 0$. Then we have $x_1, x_2, \ldots, x_k, x_{k+1} \in \text{span}\{v_1, v_2, \ldots, v_{k-1}\}$. The induction hypothesis applies directly to show that $x_1, x_2, \ldots, x_k, x_{k+1}$ is linearly dependent.

Theorem 5

If $v_1, v_2, \ldots, v_k$ is a finite list of vectors in a vector space V then every list of $k+1$ (or more) vectors from $\text{span}\{v_1, v_2, \ldots, v_k\}$ is linearly dependent.

Induction Step (continued).

- Otherwise, we may assume that $a_{k+1,k} \neq 0$.
  Consider the vectors

$$\begin{aligned} y_1 &= x_1 - \frac{a_{1,k}}{a_{k+1,k}} x_{k+1} \\ \vdots \quad & \quad \vdots \\ y_k &= x_k - \frac{a_{k,k}}{a_{k+1,k}} x_{k+1} \end{aligned}$$

  Note that $y_1, y_2, \ldots, y_k \in \text{span}\{v_1, v_2, \ldots, v_{k-1}\}$.

Theorem 5

If $v_1, v_2, \ldots, v_k$ is a finite list of vectors in a vector space V then every list of $k+1$ (or more) vectors from $\text{span}\{v_1, v_2, \ldots, v_k\}$ is linearly dependent.

Proof.

Induction Step (continued).

By induction hypothesis there are scalars $b_1, \ldots, b_k$, not all zero, such that

$$0 = b_1 y_1 + b_2 y_2 + \cdots + b_k y_k.$$

Thus

$$0 = b_1 x_1 + b_2 x_2 + \cdots + b_k x_k - c x_{k+1}$$

for some scalar c.

Therefore $x_1, x_2, \ldots, x_k, x_{k+1}$ are linearly dependent. □