Quiz 6

MATH 24 — SPRING 2014

Sample Solutions

For which triplets of real numbers a, b, c is the matrix

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 0 \end{pmatrix}$$

diagonalizable over $\mathbb{R}$? Justify your answer.

Solution. Since the matrix is upper triangular, we see immediately that its characteristic polynomial is $(1-t)^2(0-t)$ and the eigenvalues are 0 and 1. By Theorem 5.9, knowing that the characteristic polynomial of the matrix splits, the matrix is diagonalizable if and only if the geometric multiplicity and algebraic multiplicity match for each eigenvalue. The algebraic multiplicity of the eigenvalue 0 is 1. The geometric multiplicity of an eigenvalue is always at least 1 and at most the algebraic multiplicity by Theorem 5.7. So the only possibility is that $\dim(E_0) = 1$.
The algebraic multiplicity of the eigenvalue 1 is 2. By the Dimension Theorem, the geometric multiplicity

$$\dim(E_1) = 3 - \operatorname{rank}\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 1 \end{pmatrix}.$$

- If $a = 0$, then the rank is 1 since the matrix has only one nonzero column. Therefore, $\dim(E_1) = 3 - 1 = 2$.
- If $a \neq 0$ then the two nonzero colums are linearly independent since they are visibly not multiples of each other. Therefore, the rank is 2 and $\dim(E_1) = 3 - 2 = 1$.

Thus, the algebraic and geometric multiplicities of the matrix match exactly when $a = 0$. So the triplets a, b, c of real numbers for which the given matrix is diagonalizable are precisely those where $a = 0$. $\square$