Quiz 5

MATH 24 — SPRING 2014

Sample Solutions

Let F be an arbitrary field.

(A) A square matrix A is **idempotent** if $A^k = A$ for every positive integer k. Show that every idempotent matrix $A \in \mathsf{M}_{n\times n}(F)$ has determinant 0 or determinant 1.

Solution. Since determinants are multiplicative, for an idempotent matrix A we have $\det(A)^2 = \det(A^2) = \det(A)$ since $A^2 = A$. In other words, $\det(A)(\det(A) - 1) = 0$. Since a product of two elements of a field is zero precisely if one of the two elements is zero, we conclude that $\det(A) = 0$ or $\det(A) = 1$. □

Solution. We consider two cases, depending on whether A is invertible or not.

If A is invertible, then mutiplying $A^2 = A$ by A^{-1} gives $A = I$ and so $\det(A) = \det(I) = 1$.

If A is not invertible, then $\det(A) = 0$ by the Corollary of Theorem 4.7.

Either way, we see that $\det(A) = 0$ or $\det(A) = 1$. □

(B) A square matrix B is **nilpotent** if $B^k = O$ for some positive integer k. Show that every nilpotent matrix $B \in \mathsf{M}_{n\times n}(F)$ has determinant 0.

Solution. Suppose k is a positive integer such that $B^k = O$. Then

$$0 = \det(B^k) = \det(\underbrace{B \cdots B}_{k \text{ times}}) = \underbrace{\det(B) \cdots \det(B)}_{k \text{ times}} = \det(B)^k.$$

If $\det(B)$ were nonzero then $\det(B)^k$ would be nonzero too since, in any field, a product of nonzero elements is always nonzero. Therefore, we must have $\det(B) = 0$. □

Solution. It suffices to show that B is not invertible. If B were invertible, with inverse B^{-1}, then every power B^k is invertible too since

$$B^k(B^{-1})^k = \underbrace{B \cdots B}_{k \text{ times}}\underbrace{B^{-1} \cdots B^{-1}}_{k\text{-times}} = I, \quad (B^{-1})^k B^k = \underbrace{B^{-1} \cdots B^{-1}}_{k \text{ times}}\underbrace{B \cdots B}_{k\text{-times}} = I,$$

which can be seen by computing the product starting in the middle and working out way out. However, the zero matrix O is not invertible, so we cannot have $B^k = O$ when B is invertible. Therefore, if B is nilpotent then it is not invertible and therefore $\det(B) = 0$ by the Corollary of Theorem 4.7. □

Solution. By the Corollary of Theorem 4.7, it suffices to show that B is not invertible. One way to do this is to show that $\operatorname{rank}(B) \neq n$, or equivalently that $\operatorname{nullity}(B) \neq 0$ by the Dimension Theorem (in matrix formulation).

Let k be the smallest positive integer such that $B^k = O$. If $k = 1$ then $B = O$ and $\operatorname{nullity}(B) = n$. If $k > 1$, then B^{k-1} is not the zero matrix. If $v_1, v_2, \ldots, v_n$ are the columns of B^{k-1}, then $Bv_1, Bv_2, \ldots, Bv_n$ are the columns of $B^k = BB^{k-1}$. Since B^{k-1} is not the zero matrix, then $v_i \neq 0$ for some i. But since B^k is the zero matrix, $Bv_i = 0$. Thus v_i is a nonzero element of the null space of B, which shows that $\operatorname{nullity}(B) \neq 0$. □