Quiz 4

MATH 24 — SPRING 2014

Sample Solutions

Given

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & -1 \\ 1 & 3 & 4 & 3 & -2 \\ 2 & 0 & 2 & 2 & -2 \end{pmatrix}.$$

Knowing that the first, second and fourth columns of A form a basis for $\mathbb{R}^3$, write down the reduced row echelon form of A. Justify your answer. (*Hint*: See Theorem 3.16.)

Solution. Using Theorem 3.16, we can figure out the columns of the reduced row echelon form of A from left to right as follows.

- Since the first two colums of A are linearly independent, the reduced row echelon form of A must start with columns e_1 and e_2.
- By visual inspection, we see that the third column A is the sum of the first two, by part (d) of Theorem 3.16, the third column can be used to read how the third column is a linear combination of the first two. Since the first two columns are linearly independent, there is only one way to write the third column of A as a linear combination of the first two. So the third column of the reduced row echelon form of A must be $e_1 + e_2$.
- Since we are given that the first, second and fourth columns of A are linearly independent, the fourth column of the reduced row echelon form of A must be e_3.
- The last column of A must be a linear combination of the first, second and fourth columns of A. After some computation, we see that it is the sum of the first and second, minus twice the fourth. Therefore, the last column of the reduced row echelon form of A must be $e_1 + e_2 - 2e_3$ as we discussed above.

Combining this information, we obtain

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix}.$$

□

Solution. We could also compute the reduced row echelon form of A using elementary row operations. By the Corollary of Theorem 3.16, this is necessarily <u>the</u> reduced row echelon form of A.

$$\begin{pmatrix} 1 & 3 & 4 & 3 & -2 \\ 0 & 1 & 1 & 1 & -1 \\ 2 & 0 & 2 & 2 & -2 \end{pmatrix} \qquad \text{Exchange rows 1 and 2.}$$

$$\begin{pmatrix} 1 & 3 & 4 & 3 & -2 \\ 0 & 1 & 1 & 1 & -1 \\ 0 & -6 & -6 & -4 & 2 \end{pmatrix} \qquad \text{Add } -2 \text{ times row 1 to row 3.}$$

$$\begin{pmatrix} 1 & 3 & 4 & 3 & -2 \\ 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 2 & -4 \end{pmatrix} \qquad \text{Add 6 times row 2 to row 3.}$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 2 & -4 \end{pmatrix} \qquad \text{Add } -3 \text{ times row 2 to row 1.}$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix} \qquad \text{Multiply row 3 by } 1/2.$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix} \qquad \text{Add } -1 \text{ times row 3 to row 2.}$$

□

Solution. Another option is to ask your calculator or computer to find the reduced row echelon form of A. But then how do we justify our answer?

One trick is to ask our computer to find the reduced row echelon form of the augmented matrix

$$\left(\, A \mid I \, \right) = \left(\begin{array}{ccccc|ccc} 0 & 1 & 1 & 1 & -1 & 1 & 0 & 0 \\ 1 & 3 & 4 & 3 & -2 & 0 & 1 & 0 \\ 2 & 0 & 2 & 2 & -2 & 0 & 0 & 1 \end{array} \right)$$

to find

$$\left(\begin{array}{ccccc|ccc} 1 & 0 & 1 & 0 & 1 & -3 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & -2 & 1 & -1/2 \\ 0 & 0 & 0 & 1 & -2 & 3 & 1 & 1/2 \end{array} \right).$$

The augmented part

$$Q = \begin{pmatrix} -3 & 1 & 0 \\ -2 & 1 & -1/2 \\ 3 & 1 & 1/2 \end{pmatrix}$$

records the product of the elementary matrices used to obtain the reduced row echelon form of A.

Every invertible matrix is a product of elementary matrices by Corollary 3 to Theorem 3.6. If Q is invertible and QA is in reduced row echelon form, then it follows from the Corollary to Theorem 3.16 that QA is the reduced row echelon form of A. Since

$$QA = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$

is indeed in reduced row echelon form, it suffices to show that Q is invertible.

There are several ways to show that Q is invertible. A simple way is to compute the inverse

$$Q^{-1} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 3 & 3 \\ 2 & 0 & 2 \end{pmatrix}$$

and show that $Q^{-1}Q = I = QQ^{-1}$.

A clever way which avoids computing Q^{-1} is to observe that QA visibly has rank 3 since it has e_1, e_2, e_3 among its columns. Since the rank of QA cannot be larger than the rank of Q by Theorem 3.7(b), we conclude that Q has rank 3 too and therefore it must be invertible! $\square$