Homework Notes — Week 8

Math 24 — Spring 2014

§6.4#5 Suppose that T is normal. I will show that $T - cI$ is normal too. By Theorem 6.11(a,b,e), we have

$$(T - cI)^* = T^* - \bar{c}I.$$

Therefore,

$$(T - cI)(T - cI)^* = (T - cI)(T^* - \bar{c}I) = TT^* - cT^* - \bar{c}T + c\bar{c}I$$

and

$$(T - cI)^*(T - cI) = (T^* - \bar{c}I)(T - cI) = T^*T - cT^* - \bar{c}T + c\bar{c}I.$$

Since T is normal, $TT^* = T^*T$ and we immediately see that $(T - cI)$ is normal too.

§6.4#9* **Theorem.** If T is a normal operator on a finite dimensional inner product space, then $\mathsf{N}(T) = \mathsf{N}(T^*)$ and $\mathsf{R}(T) = \mathsf{R}(T^*)$.

Proof. The fact that $\mathsf{N}(T) = \mathsf{N}(T^*)$ is a consequence of Theorem 6.15(a), which says that $\|T(x)\| = \|T^*(x)\|$ for all x. Therefore,

$$T(x) = 0 \quad \text{iff} \quad \|T(x)\| = 0 \quad \text{iff} \quad \|T^*(x)\| = 0 \quad \text{iff} \quad T^*(x) = 0$$

or, equivalently,

$$x \in \mathsf{N}(T) \quad \text{if and only if} \quad x \in \mathsf{N}(T^*),$$

which is the same as saying that $\mathsf{N}(T) = \mathsf{N}(T^*)$.

From, Exercise 12 of Section 6.3, we know that $\mathsf{R}(T)^\perp = \mathsf{N}(T^*)$ and $\mathsf{R}(T^*)^\perp = \mathsf{N}(T)$. Since $(\mathsf{W}^\perp)^\perp = \mathsf{W}$ by Theorem 6.7, we see that $\mathsf{R}(T) = \mathsf{N}(T^*)^\perp = \mathsf{N}(T)^\perp = \mathsf{R}(T^*)$. □

§6.6#4* *Hint*: The fact that $I - T$ is a projection (i.e., $(I - T)^2 = I - T$) follows from part E2 of the first exam. The fact that $I - T$ is self-adjoint follows from the fact that T is self-adjoint. The fact that $I - T$ is an orthogonal projection follows from Exercise 6 of this section.

§6.6#6 Note that a projection $T : \mathsf{V} \to \mathsf{V}$ has only two possible eigenvalues: 0 and 1. Indeed, if x is a nonzero vector and λ is a scalar such that $T(x) = \lambda x$, then $T^2(x) = \lambda^2 x$ but also $T^2(x) = \lambda x$ since $T^2 = T$. Therefore, $\lambda^2 = \lambda$, which has only two solutins $\lambda_1 = 0$ and $\lambda_2 = 1$.

If T is normal, then we can apply the Spectral Theorem to find orthogonal projections $T_1 : \mathsf{V} \to \mathsf{V}$ and $T_2 : \mathsf{V} \to \mathsf{V}$ such that $T = \lambda_1 T_1 + \lambda_2 T_2$. But $\lambda_1 = 0$ and $\lambda_2 = 1$, so $T = T_2$!

§7.1#7abcd

(a) We need to show that for every positive integer k, we have $\mathsf{N}(U^k) \subseteq \mathsf{N}(U^{k+1})$. Well, if $U^k(x) = 0$ then $U^{k+1}(x) = U(U^k(x)) = U(0) = 0$.

(c) First note that the hypothesis $\text{rank}(U^{k+1}) = \text{rank}(U^k)$ implies that $\mathsf{N}(U^{k+1} = \mathsf{N}(U^k)$. Indeed, it follows from the Dimension Theorem that $\dim(\mathsf{N}(U^{k+1})) = \dim(\mathsf{N}(U^k))$ and since $\mathsf{N}(U^k) \subseteq \mathsf{N}(U^{k+1})$ by part (a), it follows that $\mathsf{N}(U^{k+1} = \mathsf{N}(U^k)$.

I will now show that if $\mathsf{N}(U^{k+1}) = \mathsf{N}(U^k)$ then $\mathsf{N}(U^n) = \mathsf{N}(U^k)$ for all positive integers $n \geq k$.

The proof is by induction on $n \geq k$. The base case ($n = k$) just says that $\mathsf{N}(U^k) = \mathsf{N}(U^k)$, which is trivially true.

For the successor step ($n \to n+1$), suppose that $\mathsf{N}(U^{k+1}) = \mathsf{N}(U^k)$ and that $\mathsf{N}(U^n) = \mathsf{N}(U^k)$. Since $U^{n+1}(x) = U^n(U(x))$, we see that

$$\mathsf{N}(U^{n+1}) = \{x \in \mathsf{V} : U(x) \in \mathsf{N}(U^n)\}.$$

Since $\mathsf{N}(U^n) = \mathsf{N}(U^k)$ by the induction hypothesis, we see that

$$\mathsf{N}(U^{n+1}) = \{x \in \mathsf{V} : U(x) \in \mathsf{N}(U^k)\} = \mathsf{N}(U^{k+1}).$$

Because $\mathsf{N}(U^{k+1}) = \mathsf{N}(U^k)$, we conclude that $\mathsf{N}(U^{n+1}) = \mathsf{N}(U^k)$.

(b) By part (a) and the Dimension Theorem, for all positive integers $n \geq k$, we have

$$\text{rank}(U^n) = \text{rank}(U^k) \quad \text{iff} \quad \text{nullity}(U^n) = \text{nullity}(U^k) \quad \text{iff} \quad \mathsf{N}(U^n) = \mathsf{N}(U^k),$$

where the last equality is because we know that $\mathsf{N}(U^k) \subseteq \mathsf{N}(U^n)$ by part (a). So part (b) follows immediately from part (c) above.

(d) By definition, $x \in \mathsf{K}_\lambda$ if and only if $x \in \mathsf{N}(T-\lambda I)^n$ for some positive integer n. By part (a) (with $U = T - \lambda I$) we have

$$\mathsf{N}(T-\lambda I) \subseteq \mathsf{N}(T-\lambda I)^2 \subseteq \mathsf{N}(T-\lambda I)^3 \subseteq \cdots$$

and, by part (c), if for any positive integer m we have

$$\mathsf{N}(T-\lambda I)^{m+1} = \mathsf{N}(T-\lambda I)^m$$

then the sequence stabilizes from that point on. So then, if $x \in \mathsf{N}(T-\lambda I)^n$, then either $n \leq m$ and $x \in \mathsf{N}(T-\lambda I)^n \subseteq \mathsf{N}(T-\lambda I)^m$, or else $n > m$ and $x \in \mathsf{N}(T-\lambda I)^n = \mathsf{N}(T-\lambda I)^m$. Therefore, $\mathsf{K}_\lambda \subseteq \mathsf{N}(T-\lambda I)^m$ and since we necessarily have $\mathsf{N}(T-\lambda I)^m \subseteq \mathsf{K}_\lambda$ by definition of K_λ, we conclude that $\mathsf{N}(T-\lambda I)^m = \mathsf{K}_\lambda$.