Homework Notes — Week 7

Math 24 — Spring 2014

§6.1#4

(a) Complete the proof in example 5 that $\langle \cdot, \cdot \rangle$ is an inner product (the Frobenius inner product) on $M_{n\times n}(F)$. In the example properties (a) and (d) have already been verified, so we need to check that (b) and (c) also hold.

To see that (b) holds, suppose that $A, B \in M_{n\times n}(F)$ and $a \in F$. Then

$$\begin{aligned}\langle aA, B\rangle &= \operatorname{tr}(B^*(aA)),\\ &= \operatorname{tr}(aB^*A),\\ &= a\operatorname{tr}(B^*A),\\ &= a\langle A, B\rangle,\end{aligned}$$

where the third equality holds by linearity of the trace, so property (c) holds.

For property (c) holds, we compute both $\langle A, B\rangle$ and $\langle B, A\rangle$,

$$\begin{aligned}\langle A, B\rangle &= \operatorname{tr}(B^*A),\\ &= \sum_{i=1}^{n}(B^*A)_{i,i},\\ &= \sum_{i=1}^{n}\sum_{k=1}^{n}(B^*)_{i,k}A_{k,i},\\ &= \sum_{i=1}^{n}\sum_{k=1}^{n}\overline{B_{k,i}}A_{k,i},\end{aligned}$$

where the final equality follows because the matrix B^* is defined by $(B^*)_{i,k} = \overline{B_{k,i}}$. An identical calculation gives that $\langle B, A\rangle = \sum_{i=1}^{n}\sum_{k=1}^{n}\overline{A_{k,i}}B_{k,i}$ and since complex conjugation has the following three properties: $\overline{z+z'} = \overline{z}+\overline{z'}$, $\overline{zz} = \overline{z}\overline{z'}$ and $\overline{\overline{z}} = z$ for all complex numbers $z, z' \in \mathbb{C}$, we have that

$$
\begin{aligned}
\overline{\langle B, A\rangle} &= \overline{\sum_{i=1}^{n}\sum_{k=1}^{n} \overline{A_{k,i}} B_{k,i}}, \\
&= \sum_{i=1}^{n}\sum_{k=1}^{n} \overline{\overline{A_{k,i}} B_{k,i}}, \\
&= \sum_{i=1}^{n}\sum_{k=1}^{n} \overline{\overline{A_{k,i}}}\,\overline{B_{k,i}}, \\
&= \sum_{i=1}^{n}\sum_{k=1}^{n} A_{k,i}\overline{B_{k,i}}, \\
&= \sum_{i=1}^{n}\sum_{k=1}^{n} \overline{B_{k,i}} A_{k,i}, \\
&= \langle A, B\rangle.
\end{aligned}
$$

(b) Use the Frobenius inner product to compute $\|A\|, \|B\|$ and $\langle A, B\rangle$ for

$$
A = \begin{pmatrix} 1 & 2+i \\ 3 & i \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1+i & 0 \\ i & -i \end{pmatrix}.
$$

By definition $||X|| = \sqrt{\langle X, X\rangle}$ for all matrices $X \in M_{n\times n}(F)$, so we star by computing $\langle A, A\rangle, \langle B, B\rangle$,

$$\begin{aligned}
A^*A &= \begin{pmatrix} 1 & 3 \\ 2-i & -i \end{pmatrix}\begin{pmatrix} 1 & 2+i \\ 3 & i \end{pmatrix}, \\
&= \begin{pmatrix} 10 & 2+4i \\ 2-4i & 6 \end{pmatrix}, \\
B^*B &= \begin{pmatrix} 1-i & -i \\ 0 & i \end{pmatrix}\begin{pmatrix} 1+i & 0 \\ i & -i \end{pmatrix}, \\
&= \begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix}, \\
B^*A &= \begin{pmatrix} 1-i & -i \\ 0 & i \end{pmatrix}\begin{pmatrix} 1 & 2+i \\ 3 & i \end{pmatrix}, \\
&= \begin{pmatrix} 1-4i & 4-i \\ 3i & 1+i \end{pmatrix}.
\end{aligned}$$

Now we are in a position to compute the various inner products involved,

$$\begin{aligned}
\langle A, A\rangle &= \operatorname{tr}(A^*A), \\
&= 10+6, \\
&= 16, \\
\langle B, B\rangle &= \operatorname{tr}(B^*B), \\
&= 3+1, \\
&= 4, \\
\langle A, B\rangle &= \operatorname{tr}(B^*A), \\
&= 1-4i+1+i, \\
&= 2-3i.
\end{aligned}$$

So finally we can compute both $\|A\|$ and $\|B\|$,

$$\|A\| = \sqrt{16} = 4, \text{ and } \|B\| = \sqrt{4} = 2.$$

§6.1#12* **Theorem.** Let V be an inner product space and let $\{v_1, \ldots, v_k\}$ be an orthogonal set in V. Then, for any scalars $a_1, \ldots, a_k$:

$$\left\| \sum_{i=1}^{k} a_i v_i \right\|^2 = \sum_{i=1}^{k} |a_i|^2 \cdot \|v_i\|^2.$$

Proof. With v_i, a_i described as above, we compute $\left\|\sum_{i=1}^k a_i v_i\right\|^2 = \left\langle \sum_{i=1}^k a_i v_i, \sum_{j=1}^k a_j v_j \right\rangle$ using the linearity in the first variable, then and conjugate-linearity in the second (Theorem 6.1), namely

$$\left\langle \sum_{i=1}^{k} a_i v_i, \sum_{j=1}^{k} a_j v_j \right\rangle = \sum_{i=1}^{k} a_i \left\langle v_i, \sum_{j=1}^{k} a_j v_j \right\rangle,$$
$$= \sum_{i=1}^{k} \sum_{j=1}^{k} a_i \overline{a_j} \langle v_i, v_j \rangle.$$

But $\langle v_i, v_j \rangle = 1$ if $j = i$ and 0 if $i \neq j$.Therefore for any fixed i with $1 \leq i \leq k$ we have

$$\left\| \sum_{i=1}^{k} a_i v_i \right\|^2 = \left\langle \sum_{i=1}^{k} a_i v_i, \sum_{j=1}^{k} a_j v_j \right\rangle,$$
$$= \sum_{i=1}^{k} \sum_{j=1}^{k} a_i \overline{a_j} \langle v_i, v_j \rangle,$$
$$= \sum_{i=1}^{k} a_i \overline{a_i} \langle v_i, v_i \rangle,$$
$$= \sum_{i=1}^{k} |a_i|^2 \|v_i\|^2.$$

□

§6.1#17 Let T be a linear operator on an inner product space V, and suppose that $\|T(x)\| = \|x\|$ for all $x \in V$. Then T is one-to-one.

Proof. We appeal to the fact that T is one-to-one if and only if $N(T) = \{0\}$ (Theorem 2.4). So suppose that $x \in N(T)$, i.e. that $T(x) = 0$. Then we must have

$$\|x\| = \|T(x)\| = \|0\| = 0,$$

but the only vector with $\|x\| = 0$ is the zero vector, so $x = 0$. So $N(T) = \{0\}$ and therefore T is one-to-one. □

§6.2#2 In each part, apply the Gram-Schmidt process to the given subset S of the inner product space V to obtain an orthogonal basis for span(S). Then normalize the vectors to obtain an orthonormal basis for span(S), and finally compute the Fourier coefficients of the given vector relative to β. Finally use Theorem 6.5 to verify the result.

(b) $V = \mathbb{R}^3, S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ and $x = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

To initialize the Gram-Schmidt process we take $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and then apply the next step to obtain v_2 as

$$v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \frac{\left\langle \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle}{\left\| \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\|^2} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

$$= \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \frac{2}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

$$= \frac{1}{3} \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}.$$

Next we obtain v_3 as

$$v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \frac{\left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle}{\left\| \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\|^2} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{\left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \frac{1}{3} \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right\rangle}{\left\| \frac{1}{3} \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right\|^2} \frac{1}{3} \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix},$$

$$= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{3}{2} \cdot \frac{1}{3} \cdot \frac{1}{3} \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix},$$

$$= \begin{pmatrix} 0 - \frac{1}{3} + \frac{1}{3} \\ 0 - \frac{1}{3} - \frac{1}{6} \\ 1 - \frac{1}{3} - \frac{1}{6} \end{pmatrix},$$

$$= \frac{1}{2} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

The set $\{v_1, v_2, v_3\}$ is an orthogonal basis, so to obtain an orthonormal basis by setting $w_i = \frac{1}{\|v_i\|} v_i$ for $i = 1, 2, 3$ then these vectors will work, specifically

$$\left\{ \frac{1}{\|v_1\|} v_1, \frac{1}{\|v_2\|} v_2, \frac{1}{\|v_3\|} v_3 \right\} = \left\{ \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \right\}$$

is an orthonormal basis for $\text{span}(S)$.

Now we compute the inner products $\langle x, w_i \rangle$ for $i = 1, 2, 3$,

$$
\begin{aligned}
\langle x, w_1 \rangle &= \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle, \\
&= 1 \cdot \frac{1}{\sqrt{3}} + 0 \cdot \frac{1}{\sqrt{3}} + 1 \cdot \frac{1}{\sqrt{3}}, \\
&= \frac{2}{\sqrt{3}}, \\
\langle x, w_2 \rangle &= \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right\rangle, \\
&= 1 \cdot \frac{-2}{\sqrt{6}} + 0 \cdot \frac{1}{\sqrt{6}} + 1 \cdot \frac{1}{\sqrt{6}}, \\
&= \frac{-1}{\sqrt{6}}, \\
\langle x, w_3 \rangle &= \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\rangle, \\
&= 1 \cdot 0 + 0 \cdot \frac{-1}{\sqrt{2}} + 1 \cdot \frac{1}{\sqrt{2}}, \\
&= \frac{1}{\sqrt{2}}.
\end{aligned}
$$

Now we are supposed to verify Theorem 6.5, specifically to check that

$$x = \langle x, w_1 \rangle w_1 + \langle x, w_2 \rangle w_2 + \langle x, w_3 \rangle w_3.$$

Well we can commute the right hand side of this equation directly,

$$\langle x, w_1\rangle w_1 + \langle x, w_2\rangle w_2 + \langle x, w_3\rangle w_3 = \frac{2}{\sqrt{3}}\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix} + \frac{1}{\sqrt{6}}\begin{pmatrix}-2\\1\\1\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}0\\-1\\1\end{pmatrix},$$

$$= \begin{pmatrix}\frac{2}{3}+\frac{2}{6}+0\\ \frac{2}{3}-\frac{1}{6}-\frac{1}{2}\\ \frac{2}{3}-\frac{1}{6}+\frac{1}{2}\end{pmatrix},$$

$$= \begin{pmatrix}1\\0\\1\end{pmatrix},$$

$$= x.$$

(d) $V = \text{span}(S)$ where $S = \left\{\begin{pmatrix}1\\i\\0\end{pmatrix}, \begin{pmatrix}1-i\\2\\4i\end{pmatrix}\right\}$ and $x = \begin{pmatrix}3+i\\4i\\-4\end{pmatrix}$.

To start we set $v_1 = \begin{pmatrix}1\\i\\0\end{pmatrix}$ and then compute v_2 by

$$v_2 = \begin{pmatrix}1-i\\2\\4i\end{pmatrix} - \frac{\left\langle \begin{pmatrix}1-i\\2\\4i\end{pmatrix}, \begin{pmatrix}1\\i\\0\end{pmatrix}\right\rangle}{\left\langle \begin{pmatrix}1\\i\\0\end{pmatrix}, \begin{pmatrix}1\\i\\0\end{pmatrix}\right\rangle}\begin{pmatrix}1\\i\\0\end{pmatrix},$$

$$= \begin{pmatrix}1-i\\2\\4i\end{pmatrix} - \frac{(1-i)\cdot\overline{1} + 2\cdot\overline{i} + 4i\cdot\overline{0}}{1\cdot\overline{1} + i\cdot\overline{i} + 0\cdot\overline{0}}\begin{pmatrix}1\\i\\0\end{pmatrix},$$

$$= \begin{pmatrix}1-i\\2\\4i\end{pmatrix} - \frac{1-i-2i}{1-i^2}\begin{pmatrix}1\\i\\0\end{pmatrix},$$

$$= \begin{pmatrix}1-i\\2\\4i\end{pmatrix} - \frac{1-3i}{2}\begin{pmatrix}1\\i\\0\end{pmatrix},$$

$$= \begin{pmatrix} 1 - i - \frac{1}{2}(1-3i) \\ 2 - \frac{1}{2}(i - 3i^2) \\ 4i \end{pmatrix},$$

$$= \begin{pmatrix} \frac{1}{2} + \frac{1}{2}i \\ \frac{1}{2} - \frac{1}{2}i \\ 4i \end{pmatrix},$$

$$= \frac{1}{2}\begin{pmatrix} 1+i \\ 1-i \\ 8i \end{pmatrix}.$$

If we now normalize the vectors v_1, v_2 we get $w_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}$ and $w_2 = \frac{1}{2\sqrt{17}}\begin{pmatrix} 1+i \\ 1-i \\ 8i \end{pmatrix}$.

Now we compute $\langle x, w_1 \rangle$ and $\langle x, w_2 \rangle$,

$$\langle x, w_1 \rangle = \left\langle \begin{pmatrix} 3+i \\ 4i \\ -4 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \right\rangle,$$

$$= \frac{1}{\sqrt{2}}(3 + i \cdot \overline{1} + 4i \cdot \overline{i} + (-4) \cdot 0),$$

$$= \frac{1}{\sqrt{2}}(3 + i + 4),$$

$$= \frac{1}{\sqrt{2}}(7 + i),$$

$$\langle x, w_2 \rangle = \left\langle \begin{pmatrix} 3+i \\ 4i \\ -4 \end{pmatrix}, \frac{1}{2\sqrt{17}}\begin{pmatrix} 1+i \\ 1-i \\ 8i \end{pmatrix} \right\rangle,$$

$$= \frac{1}{2\sqrt{17}}((3+i) \cdot (\overline{1+i}) + 4i \cdot (\overline{1-i}) - 4 \cdot \overline{8i}),$$

$$= \frac{1}{2\sqrt{17}}((3+i)(1-i) + 4i(1+i) + 32i),$$

$$
\begin{aligned}
&= \frac{1}{2\sqrt{17}}(3 - 3i + i - i^2 + 4i + 4i^2 + 32i), \\
&= \frac{1}{2\sqrt{17}} \cdot 34i, \\
&= \frac{34i}{2\sqrt{17}}.
\end{aligned}
$$

Now again we check that $x = \langle x, w_1 \rangle w_1 + \langle x, w_2 \rangle w_2$ in accordance with Theorem 6.5,

$$
\begin{aligned}
\langle x, w_1 \rangle w_1 + \langle x, w_2 \rangle w_2 &= \frac{1}{\sqrt{2}}(7+i)\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} + \frac{34i}{2\sqrt{17}} \frac{1}{2\sqrt{17}} \begin{pmatrix} 1+i \\ 1-i \\ 8i \end{pmatrix}, \\
&= \frac{7+i}{2} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} + \frac{34i}{2 \cdot 34} \begin{pmatrix} 1+i \\ 1-i \\ 8i \end{pmatrix}, \\
&= \frac{1}{2} \begin{pmatrix} 7+i \\ -1+7i \\ 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -1+i \\ 1+i \\ -8 \end{pmatrix}, \\
&= \frac{1}{2} \begin{pmatrix} 7+i-1+i \\ -1+7i+1+i \\ -8 \end{pmatrix}, \\
&= \frac{1}{2} \begin{pmatrix} 6+2i \\ 8i \\ -8 \end{pmatrix}, \\
&= \begin{pmatrix} 3+i \\ 4i \\ -4 \end{pmatrix}, \\
&= x.
\end{aligned}
$$

§6.2#10 Let $W = \text{span}\left\{ \begin{pmatrix} i \\ 0 \\ 1 \end{pmatrix} \right\}$ in $\mathbb{C}^3$. Find orthonormal bases for W and $W^\perp$. The set $\left\{ \begin{pmatrix} i \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis for W by construction, hence by normalizing this vector we obtain an orthonormal basis of W, namely $\left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 0 \\ 1 \end{pmatrix} \right\}$.

a vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{C}^3$ is in $W^\perp$ if and only if

$$0 = \left\langle \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} i \\ 0 \\ 1 \end{pmatrix} \right\rangle = -ix + z,$$

so two such vectors are $\begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. These two vectors are orthogonal, linearly independent, and span $W^\perp$ so that $\left\{ \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ is a basis for $W^\perp$. Normalizing these vectors we obtain the orthonormal basis of $W^\perp$

$$\left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

§6.3#2 For each oft he following inner product space V (over F) and linear transformations $g : V \to F$, find a vector y such that $g(x) = \langle x, y \rangle$ for all $x \in V$

(b) $C = \mathbb{C}^2, g \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = z_1 - z_2$.

Let $y = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, then for any $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{C}^2$ we have

$$\begin{aligned} \left\langle \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\rangle &= z_1 \cdot \overline{1} + z_2 \cdot (\overline{-2}), \\ &= z_1 - 2z_2, \\ &= g \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}. \end{aligned}$$

§6.3#3 For each of the following inner product space V and linear operators T on V, evaluate T^* at the given vector in V.

(b) $V = \mathbb{C}^2, T\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 2z_1 + iz_2 \\ (1-i)z_1 \end{pmatrix}$ and $x = \begin{pmatrix} 3-i \\ 1+2i \end{pmatrix}$.

First note that $T\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 2 & i \\ 1-i & 0 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ for all vectors $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{C}^2$. Let $\beta = \left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$, i.e. the standard basis of $\mathbb{C}^2$, then $[T]_\beta = \begin{pmatrix} 2 & i \\ 1-i & 0 \end{pmatrix}$. Since $[T^*]_\beta = [T]_\beta^*$ we have that

$$[T^*]_\beta = \begin{pmatrix} 2 & i \\ 1-i & 0 \end{pmatrix}^* = \begin{pmatrix} 2 & 1+i \\ -i & 0 \end{pmatrix}.$$

Now we remark that for any $v \in \mathbb{C}$ we have $v = [v]_\beta$, so

$$\begin{aligned} T^*\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} &= \left[T^*\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right]_\beta, \\ &= [T^*]_\beta \left[\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right]_\beta, \\ &= \begin{pmatrix} 2 & 1+i \\ -i & 0 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}. \end{aligned}$$

So we can compute T^* on all vectors of $\mathbb{C}^2$ by the above matrix formula, therefore

$$\begin{aligned} T\begin{pmatrix} 3-i \\ 1+2i \end{pmatrix} &= \begin{pmatrix} 2 & 1+i \\ -i & 0 \end{pmatrix}\begin{pmatrix} 3-i \\ 1+2i \end{pmatrix}, \\ &= \begin{pmatrix} 2\cdot(3-i) + (1+i)\cdot(1+2i) \\ -i\cdot(3-i) + 0\cdot(1+2i) \end{pmatrix}, \\ &= \begin{pmatrix} 6 - 2i + 1 + 2i + i + 2i^2 \\ -3i + i^2 \end{pmatrix}, \\ &= \begin{pmatrix} 5+i \\ -1-3i \end{pmatrix}, \end{aligned}$$

§6.3#12a* **Theorem.** Let V be an inner product space and let T be a linear operator on V. Then $R(T^*)^\perp = N(T)$.

Proof. We will show that following list of statements are all equivalent,

(i) $x \in N(T)$,

(ii) $T(x) = 0$,

(iii) $\langle T(x), y\rangle = 0$ for all $y \in V$,

(iv) $\langle x, T^*(y)\rangle = 0$ for all $y \in V$,

(v) $\langle x, w\rangle = 0$ for all $w \in R(T^*)$,

(vi) $x \in R(T^*)^\perp$,

so in particular once we've done this we have that $N(T) = R(T^*)^\perp$ by the equivalence of (i) and (vi).

That (i) and (ii) are equivalent is just the definition of $N(T)$. That (ii) and (iii) are equivalent is just the observation that the only vector $u \in V$ with $\langle u, v\rangle = 0$ for all $v \in V$ is $u = 0$.

That (iii) and (iv) are equivalent is just the fact that for any vectors $u, v \in V$ we have $\langle T(x), y\rangle = \langle x, T^*(y)\rangle$ since T^* is the adjoint of T and this property is the defining property of the adjoint.

The equivalence of (iv) and (v) follows form the fact that $w \in R(T^*)$ if and only if $w = T(y)$ for some $y \in V$.

Finally the equivalence of (v) and (vi) is just the definition of $R(T^*)^\perp$. So (i) and (vi) are equivalent, which means exactly that $R(T^*)^\perp = N(T)$.

□