Homework Notes — Week 5

Math 24 — Spring 2014

§3.1#8* **Theorem.** *If a matrix Q can be obtained from a a matrix P by an elementary row operation, then P can be obtained from Q by an elementary row operation of the same type.*

Proof. There are three types of elementary row operations and we treat them separately.

Type 1. Suppose P is obtained from Q by interchanging two rows.

If this is the case, say that P is obtained from Q by interchanging rows i and j of Q. Then the matrix obtained by interchanging rows i and j of P is Q so that Q is obtained from P by interchanging two rows.

Type 2. Suppose P is obtained from Q by multiplying a row of Q by a nonzero scalar.

Say that P is obtained from Q by multiplying row i of Q by the nonzero scalar c. Then the matrix obtained from multiplying row i of P by the nonzero scalar c^{-1} is the matrix Q. So Q is obtained from P my multiplying a row by a nonzero scalar.

Type 3. Suppose P is obtained from Q by adding a scalar multiple of a row of Q to another row of Q.

Say that P is obtained from Q by adding c times row i of Q to row j of Q. Then the matrix obtained from adding $-c$ times row i of P to row j of P is precisely Q. Therefore Q is obtained from P by adding a scalar multiple of a row of P to another row of P. □

§3.2#6bf For (b), we are asked to determine whether $T(f(x)) = (x+1)f'(x)$ is invertible. Since the derivative of a constant polynomial is zero, $1 \in \mathsf{N}(T)$ and therefore T cannot be invertible since it is not one-to-one. We can also check this by computing $[T]_\beta$ with respect to the standard ordered basis $\beta = \{1, x, x^2\}$ for $\mathsf{P}_2(\mathbb{R})$. Since $T(1) = 0$, $T(x) = x + 1$, $T(x^2) = (x+1)(2x) = 2x^2 + 2x$, we see that

$$[T]_\beta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix},$$

which visibly has rank at most 2 by Corollary 2(b) of Theorem 3.6 since the first column is 0. Therefore T cannot be invertible by the remark following the definition of rank on page 152.

For (f), we are asked to determine whether

$$T(A) = (\operatorname{tr}(A), \operatorname{tr}(A^t), \operatorname{tr}(EA), \operatorname{tr}(AE))$$

is invertible, where $E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Let's compute $[T]_\alpha^\gamma$, where $\alpha = \{E^{11}, E^{12}, E^{21}, E^{22}\}$ is the standard basis of $\mathsf{M}_{2\times 2}(\mathbb{R})$ $\gamma = \{e_1, e_2, e_3, e_4\}$ is the standard ordered basis for $\mathbb{R}^4$. To this end, it helps to note that EA exchanges the two rows of A and AE exhanges the two columns of A. Thus

$$\begin{aligned} T(E^{11}) &= (\operatorname{tr}(E^{11}), \operatorname{tr}(E^{11}), \operatorname{tr}(E^{21}), \operatorname{tr}(E^{12})) = (1,1,0,0), \\ T(E^{12}) &= (\operatorname{tr}(E^{12}), \operatorname{tr}(E^{21}), \operatorname{tr}(E^{22}), \operatorname{tr}(E^{11})) = (0,0,1,1), \\ T(E^{21}) &= (\operatorname{tr}(E^{21}), \operatorname{tr}(E^{12}), \operatorname{tr}(E^{11}), \operatorname{tr}(E^{22})) = (0,0,1,1), \\ T(E^{22}) &= (\operatorname{tr}(E^{22}), \operatorname{tr}(E^{22}), \operatorname{tr}(E^{12}), \operatorname{tr}(E^{21})) = (1,1,0,0), \end{aligned}$$

and hence

$$[T]_\alpha^\gamma = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

This matrix visibly has rank at most 2 by Corollary 2(b) of Theorem 3.6 since it only has two distinct columns. Therefore T cannot be invertible by the remark following the definition of rank on page 152.

§3.2#13b* **Theorem.** *The rank of any matrix equals the dimension of the subspace generated by its rows.*

Proof. Let A be any matrix. By Corollary 2(a) of Theorem 2.6, $\operatorname{rank}(A) = \operatorname{rank}(A^t)$. Since the rows of A are the columns of the transpose A^t, it follows from Theorem 3.5 that $\operatorname{rank}(A) = \operatorname{rank}(A^t)$ is the dimension of the subspace generated by the rows of A. □

§3.2#19 **Theorem.** *If $A \in \mathsf{M}_{m\times n}(F)$ has rank m and $B \in \mathsf{M}_{n\times p}(F)$ has rank n, then $AB \in \mathsf{M}_{m\times p}(F)$ has rank m.*

Proof. By definition, $\text{rank}(A) = \text{rank}(L_A)$ and $\text{rank}(B) = \text{rank}(L_B)$, where $L_A : F^n \to F^m$ and $L_B : F^p \to F^n$ are the associated left multiplication transformations. We are asked to compute $\text{rank}(AB) = \text{rank}(L_{AB}) = \text{rank}(L_A L_B)$ (see Theorem 2.15(e)) given that $\text{rank}(A) = m$ and $\text{rank}(B) = n$. To say that $\text{rank}(L_A) = m$ means that L_A is onto; to say that $\text{rank}(L_A) = n$ means that L_A is onto. Since the composition of two onto functions is onto, we see that $\text{rank}(L_A L_B) = m$. Therefore $\text{rank}(AB) = m$. □

§3.3#7be The system (b) has the immediately obvious solution $x_1 = 1, x_2 = 0, x_3 = 0$. For the system (e), if we add 3 times the first equation from the last and then subtract 2 times the second equation to the last, we obtain $0x_1 + 0x_2 + 0x_3 = 1$. Since that equation is clearly unsolvable, the system (e) cannot have a solution.

§3.3#10 **Theorem.** *If the coefficient matrix of a system of m linear equations with n unknowns has rank m, then the system has a solution.*

Proof. Let A be the $m \times n$ coefficient matrix of a system of m linear equations with n unknowns and suppose that $\text{rank}(A) = m$. To see that the system $Ax = b$ always has a solution, first recall that $\text{rank}(A) = \text{rank}(L_A) = m$ where $L_A : F^n \to F^m$ is left multiplication by A. Since $\dim(\mathsf{R}(L_A)) = \text{rank}(L_A) = m = \dim(F^m)$, we must have that L_A is onto. By definition of onto, for every $b \in F^m$ there is an $x \in F^n$ such that $b = L_A(x) = Ax$. In other words, for every $b \in F^m$, the system $Ax = b$ does have a solution. □