Homework Notes — Week 1

MATH 24 — SPRING 2014

§**1.2#9*** The most common error for this problem was to use Theorem 1.1 in a non-literal manner. Theorem 1.1 gives only one of the four possible cancellation laws:

1. If $x + z = y + z$ then $x = y$.
2. If $z + x = z + y$ then $x = y$.
3. If $x + z = z + y$ then $x = y$.
4. If $z + x = y + z$ then $x = y$.

To obtain the other laws, you need to use commutativity (VS 1). Since this section of the book is about use of the rules (VS 1–8), it is important to mention each use of these rules!

To avoid repetition, you can state and prove the relevant fact and reference that instead:

> *A useful corollary of Theorem 1.1 is the following.*
>
> ***Corollary 0.*** *If x, y and z are vectors in a vector space V such that $z + x = z + y$, then $x = y$.*
>
> *Proof.* By applying (VS 1) to both sides of $z + x = z + y$, we obtain $x + z = y + z$ and then $x = y$ follows by Theorem 1.1. □

After then, you can freely use Corollary 0 in your proofs. (I called the result Corollary 0 to emphasize that this result does not depend on Corollary 1 nor Corollary 2, so it is not circular to use it in these proofs.)

After section 1.2, it is no longer necessary to mention every use of the rules (VS 1–8) since they are no longer the main topic of discussion. You can freely use commutativity and associativity of addition without pointing out this fact every time. However, you still need to mention them when they are key elements of discussion, such as when proving that something is a vector space from scratch (as in Theorem 2.7(b), for example).

§**1.2#18** For this problem, keep in mind that

$$(a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 + 3b_2)$$

is now the *definition* of "+" and the usual addition is not meaningful in the context of this problem.

In addition to counterexamples for (VS 1–8), a counterexample for any property which is known to hold in all vector spaces can be used as a way to show that this V is not a vector space. For example, in any vector space we have that

$$2x = (1+1)x = 1x + 1x = x + x$$

holds for every vector x, but in V we have

$$2(a_1, a_2) = (2a_1, 2a_2) \quad \text{and} \quad (a_1, a_2) + (a_1, a_2) = (3a_1, 4a_2).$$

Choosing $a_1 = a_2 = 1$, say, gives a counterexample.

§**1.3#10** To show that W_1 is a subspace, you need to *prove* all three criteria of Theorem 1.3. To make sure your proofs are complete, it can be helpful to restate them as theorems.

> ***Theorem.*** *If* $(a_1, a_2, \ldots, a_n)$ *and* $(b_1, b_2, \ldots, b_n)$ *are in* W_1, *then the sum* $(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n)$ *is also in* W_1.

Even if you only do this as a mental exercise and you write your argument in a more succinct way, it will help you remember to properly introduce your notation and to include all steps (such as properly expanding definitions) which are necessary for a complete proof. Here is an example where I omitted the statement above and replaced it by a reference to Theorem 1.3(b).

> *To see that condition (b) of Theorem 1.3 holds for* W_1, *suppose that* $(a_1, \ldots, a_n)$ *and* $(b_1, \ldots, b_n)$ *are in* W_1. *Then, by definition of* W_1,
>
> $$a_1 + \cdots + a_n = 0 = b_1 + \cdots + b_n.$$

Therefore,

$$(a_1 + b_1) + \cdots + (a_n + b_n) = (a_1 + \cdots + a_n) + (b_1 + \cdots + b_n) = 0,$$

which shows that the sum

$$(a_1, \dots, a_n) + (b_1, \dots, b_n) = (a_1 + b_1, \dots, a_n + b_n)$$

is also in W_1.

§**1.3#13** The main difficulty for this problem is understanding what this space $\mathcal{F}(S, F)$ actually is. This is an important example to understand since the majority of vector spaces used in mathematics are subspaces of $\mathcal{F}(S, \mathbb{R})$ or $\mathcal{F}(S, \mathbb{C})$, for some choice of set S.

The vectors in the space $\mathcal{F}(S, F)$ are all possible functions $f : S \to F$, i.e., all possible ways to assign values in F to every point of the set S. For example, $\mathcal{F}(\mathbb{R}, \mathbb{R})$ contains most of the functions $f : \mathbb{R} \to \mathbb{R}$ from calculus, such as

$$f(x) = x^2, \quad g(x) = \sin(x), \quad \text{and} \quad h(x) = e^x.$$

In the general case, S can be any fixed set, such as

$$\mathbb{R}, \quad \mathbb{Z}, \quad [0, 1], \quad \{1, 1/2, 1/3, \dots\},$$

and even sets of things other than numbers such as

$$\{\text{Ann}, \text{Bob}, \text{Sue}, \text{Zak}\}.$$

The subspace in question

$$\mathsf{W} = \{f \in \mathcal{F}(S, F) : f(s_0) = 0\}$$

consists of all functions $f : S \to F$ that take the value 0 at the designated point s_0 of S. The point s_0 is arbitrary but it is a *fixed* element of S. For example, we could have $S = F = \mathbb{R}$ as above and $s_0 = \pi$, in which case $\sin(x)$ would be in W but $\cos(x)$ would not be in W.

Now that we understand what $\mathcal{F}(S, F)$ and W consist of, it is easy to check the three properties of Theorem 1.3:

(a) *The zero element of* $\mathcal{F}(S, F)$ *is the constant function with value* 0. *Since this function takes value* 0 *at every point of* S, *in particular at* s_0, *it does belong to* W.

(b) *If* f, g *are functions in* W *then* $f(s_0) = 0 = g(s_0)$. *Therefore*

$$(f+g)(s_0) = f(s_0) + g(s_0) = 0 + 0 = 0,$$

which shows that the sum $f + g$ *is in* W *too.*

(c) *If* f *is a function in* W *then* $f(s_0) = 0$. *Therefore, if* $c \in F$ *is any scalar, then*

$$(cf)(s_0) = c(f(s_0)) = c0 = 0,$$

which shows that the scalar product cf *is in* W *too.*

§**1.3#19*** Since this is an equivalence, we need to prove two implications:

- If $\mathsf{W}_1 \subseteq \mathsf{W}_2$ or $\mathsf{W}_2 \subseteq \mathsf{W}_1$ then $\mathsf{W}_1 \cup \mathsf{W}_2$ is a subspace.
- If $\mathsf{W}_1 \cup \mathsf{W}_2$ is a subspace then $\mathsf{W}_1 \subseteq \mathsf{W}_2$ or $\mathsf{W}_2 \subseteq \mathsf{W}_1$.

The first is clear since $\mathsf{W}_1 \cup \mathsf{W}_2 = \mathsf{W}_2$ in the first alternative and $\mathsf{W}_1 \cup \mathsf{W}_2 = \mathsf{W}_1$ in the second alternative.

For the second implication, it is easier to prove the contrapositive: if $\mathsf{W}_1 \not\subseteq \mathsf{W}_2$ and $\mathsf{W}_2 \not\subseteq \mathsf{W}_1$ then $\mathsf{W}_1 \cup \mathsf{W}_2$ is not a subspace, in fact this set is not closed under addition. To show this we need to find two vectors $x_1, x_2 \in \mathsf{W}_1 \cup \mathsf{W}_2$ such that $x_1 + x_2 \notin \mathsf{W}_1 \cup \mathsf{W}_2$. To achieve this goal x_1, x_2 can't both be in W_1 (for then $x_1 + x_2 \in \mathsf{W}_1$ too) nor can they both be in W_2 (for then $x_1 + x_2 \in \mathsf{W}_2$ too). So we must choose x_1, x_2 very carefully.

The two hypotheses $\mathsf{W}_1 \not\subseteq \mathsf{W}_2$ and $\mathsf{W}_2 \not\subseteq \mathsf{W}_1$ give us a clue how to pick x_1, x_2. The hypothesis $\mathsf{W}_1 \not\subseteq \mathsf{W}_2$ tells us that there is an $x_1 \in \mathsf{W}_1$ such that $x_1 \notin \mathsf{W}_2$. Similarly, the hypothesis $\mathsf{W}_2 \not\subseteq \mathsf{W}_1$ tells us that there is an $x_2 \in \mathsf{W}_2$ such that $x_2 \notin \mathsf{W}_1$. Any two such vectors are as required: $x_1 + x_2 \notin \mathsf{W}_1 \cup \mathsf{W}_2$. To see this, we need to show that it is impossible that either $x_1 + x_2 \in \mathsf{W}_1$ or $x_1 + x_2 \in \mathsf{W}_2$. If $x_1 + x_2 \in \mathsf{W}_1$ then

$$x_2 = (-x_1) + (x_1 + x_2) \in \mathsf{W}_1$$

since $-x_1 \in \mathsf{W}_1$; this contradicts the fact that $x_2 \notin \mathsf{W}_1$. Similarly, $x_1 + x_2 \in \mathsf{W}_2$ leads to a contradiction of the fact that $x_1 \notin \mathsf{W}_2$. We therefore conclude that $x_1 + x_2 \notin \mathsf{W}_1 \cup \mathsf{W}_2$.