Take Home Exam 1

Sample Solutions

Problem A. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation defined by

$$T(x_1, x_2, x_3, x_4) = \begin{pmatrix} x_1 + x_3 - 2x_4 \\ 2x_2 - x_3 + x_4 \\ 2x_1 + 2x_2 + x_3 - 3x_4 \end{pmatrix}.$$

(1) Find a basis for the null space $\mathsf{N}(T)$.

Solution — The equation $T(x_1, x_2, x_3, x_4) = 0$ leads to the system of equations

$$\begin{aligned} x_1 + x_3 - 2x_4 &= 0 \\ 2x_2 - x_3 + x_4 &= 0 \\ 2x_1 + 2x_2 + x_3 - 3x_4 &= 0 \end{aligned}$$

Since the third equation is 2 times the first plus the second, this system is equivalent to

$$\begin{aligned} x_1 &= -x_3 + 2x_4 \\ 2x_2 &= x_3 - x_4. \end{aligned}$$

Setting, $x_3 = s$ and $x_4 = t$, the general solution is

$$\begin{pmatrix} -s + 2t \\ \frac{1}{2}s - \frac{1}{2}t \\ s \\ t \end{pmatrix} = s \begin{pmatrix} -1 \\ 1/2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1/2 \\ 0 \\ 1 \end{pmatrix}.$$

Looking at the last two coordinates, we see that the generating vectors $v_1 = (-1, 1/2, 1, 0)$ and $v_2 = (2, -1/2, 0, 1)$ are linearly independent. Therefore $\{v_1, v_2\}$ is a basis for $\mathsf{N}(T)$.

(2) Find a basis for the range space $\mathsf{R}(T)$.

Solution — From the Claim in the proof of the Dimension Theorem from the April 9 slides, we know that if we extend the basis for $\mathsf{N}(T)$ from part 1 to a basis $\{v_1, v_2, v_3, v_4\}$ for $\mathbb{R}^4$, then $\{T(v_3), T(v_4)\}$ will be a basis for $\mathsf{R}(T)$.

The vectors $\{v_1, v_2, v_3, v_4\}$ where $v_1 = (-1, 1/2, 1, 0)$, $v_2 = (2, -1/2, 0, 1)$, $v_3 = (0, 1, 0, 0)$, $v_4 = (1, 0, 0, 0)$ span $\mathbb{R}^4$ since

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = c \begin{pmatrix} -1 \\ 1/2 \\ 1 \\ 0 \end{pmatrix} + d \begin{pmatrix} 2 \\ -1/2 \\ 0 \\ 1 \end{pmatrix} + \frac{2b - c + d}{2} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + (a + c - 2d) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

for any choice of scalars a, b, c, d. Since $\mathbb{R}^4$ has dimension 4, it follows from Corollary 2(a) of Theorem 1.10 that $\{v_1, v_2, v_3, v_4\}$ is a basis for $\mathbb{R}^4$. By the Claim mentioned above, the vectors

$$T(v_3) = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}, \quad T(v_4) = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix},$$

form a basis for $\mathsf{R}(T)$.

Problem B.

(1) Find a basis for the subspace

$$\text{span}\left\{ \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ -1 & -3 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 4 & -2 \end{pmatrix}, \begin{pmatrix} 2 & -1 \\ 2 & -2 \end{pmatrix} \right\}$$

of $\mathsf{M}_{2\times 2}(\mathbb{R})$.

Solution — We proceed from left to right, eliminating elements that are linear combinations of the previous ones. Since

$$\begin{pmatrix} 0 & 2 \\ -1 & -3 \end{pmatrix} = -3 \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 3 & -1 \\ -1 & 0 \end{pmatrix}$$

and

$$7 \begin{pmatrix} 2 & -1 \\ 2 & -2 \end{pmatrix} = -4 \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} + 6 \begin{pmatrix} 3 & -1 \\ -1 & 0 \end{pmatrix} + 5 \begin{pmatrix} 0 & -1 \\ 4 & -2 \end{pmatrix},$$

we eliminate the third and fifth matrices to obtain the set

$$S = \left\{ \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 4 & -2 \end{pmatrix} \right\}.$$

This set is linearly independent since

$$a \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} + b \begin{pmatrix} 3 & -1 \\ -1 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & -1 \\ 4 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Leads to the equations $a = -3b = 2c$ and $b = 4c$, which together imply that $a = b = c = 0$.

Since, as we have verified above, the five given matrices are in $\text{span}(S)$, we conclude that S is a basis for the given subspace of $\mathsf{M}_{2\times 2}(\mathbb{R})$.

(2) Suppose $\alpha = \{x_1, x_2, x_3\}$ and $\beta = \{y_1, y_2, y_3\}$ are two ordered bases for a vector space V over the field $\mathbb{R}$ of real numbers. Given that

$$x_1 = 2y_2 + y_3, \quad x_2 = 2y_1 + 2y_3, \quad x_2 - x_3 = y_1 - y_2,$$

find the change of coordinate matrix Q that converts α-coordinates into β-coordinates as well as the change of coordinate matrix Q^{-1} that converts β-coordinates into α-coordinates.

Solution — Since $x_1 = 2y_2 + y_3$, $x_2 = 2y_1 + 2y_3$, and

$$x_3 = x_2 - y_1 + y_2 = y_1 + y_2 + y_3,$$

we see that

$$[x_1]_\beta = (0, 2, 1), \quad [x_2]_\beta = (2, 0, 2), \quad [x_3]_\beta = (1, 1, 1).$$

Therefore,

$$Q = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

is the matrix that converts α-coordinates into β-coordinates.

Solving the given equations for y_1, y_2, y_3 in terms of x_1, x_2, x_3, we obtain

$$y_1 = -x_1 - \tfrac{1}{2}x_2 + 2x_3, \quad y_2 = -\tfrac{1}{2}x_2 + x_3, \quad y_3 = x_1 + x_2 - 2x_3.$$

Therefore

$$[y_1]_\alpha = (-1, -1/2, 2), \quad [y_2]_\alpha = (0, -1/2, 1), \quad [y_3]_\alpha = (1, 1, -2)$$

and hence

$$Q^{-1} = \begin{pmatrix} -1 & 0 & 1 \\ -1/2 & -1/2 & 1 \\ 2 & 1 & -2 \end{pmatrix}$$

is the matrix that converts β-coordinates into α-coordinates.

To be sure, we can check that $QQ^{-1} = I = Q^{-1}Q$.

(3) Let $T : \mathsf{P}_2(\mathbb{R}) \to \mathsf{P}_2(\mathbb{R})$ be the linear transformation such that

$$T(1) = 2x, \quad T(x+1) = 0, \quad T(x^2 + x + 1) = x^2 + x + 1.$$

Find the matrix representation $[T]_\gamma$ with respect to the standard ordered basis $\gamma = \{1, x, x^2\}$ for $\mathsf{P}_2(\mathbb{R})$.

Solution — From the given information and the fact that T is linear, we have $T(1) = 2x$,

$$T(x) = T(x+1) - T(1) = -2x,$$

and

$$T(x^2) = T(x^2 + x + 1) - T(x+1) = x^2 + x + 1.$$

Therefore, after reading the γ-coordinates of each result,

$$[T]_\gamma = [T]_\gamma^\gamma = \begin{pmatrix} 0 & 0 & 1 \\ 2 & -2 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Problem C.

(1) Show that an $n \times n$ matrix A over the field F is invertible if and only if its columns form a basis for F^n.

Solution — There are several ways to go about this; the following are just one of hundreds of possible solutions you could do using just the material from Chapters 1 & 2.

First, suppose that A is invertible. We will show that the columns $v_1, v_2, \ldots, v_n$ of A are linearly independent. Observe that, by definition of matrix multiplication,

$$Ax = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n,$$

where $x = (x_1, x_2, \ldots, x_n)$ is any vector in F^n. Therefore, we can interpret Ax as a linear combination of the columns of A with coefficients $x_1, x_2, \ldots, x_n$. If A is invertible with inverse A^{-1}, then $Ax = 0$ implies that $x = A^{-1}Ax = A^{-1}0 = 0$. Therefore, the only linear combination of the columns of A that yields 0 is the trivial one. Thus, if A is invertible then the columns of A are linearly independent. Since A has n columns and $\dim(F^n) = n$, it follows from Corollary 2(a) of Theorem 1.10 that $\{v_1, v_2, \ldots, v_n\}$ form a basis for F^n.

Next, suppose the columns of A form an ordered basis $\beta = \{v_1, v_2, \ldots, v_n\}$ for F^n. Note that $A = [I_{F^n}]_\beta^\alpha$, where $\alpha = \{e_1, e_2, \ldots, e_n\}$ is the standard ordered basis for F^n. Thus A is the change-of-coordinate matrix from β to α. By Theorem 2.22(a), the matrix A must be invertible. In fact, $A^{-1} = [I_{F^n}]_\alpha^\beta$.

(2) Determine all possible $a, b \in \mathbb{R}$ for which the matrix

$$\begin{pmatrix} a & 1 & 2 \\ 0 & 1 & b \\ 1 & 1 & 2 \end{pmatrix}$$

is not invertible.

Solution — If $b = 2$ (and a is any real number) then the last column is twice the second and hence the matrix is not invertible by part 1.

If $b \neq 2$, then the span of the last two columns is

$$\mathsf{W} = \{(x, y, z) \in \mathbb{R}^3 : x = z\}$$

since

$$\begin{pmatrix} x \\ y \\ x \end{pmatrix} = \frac{2y - bx}{2 - b} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{x - y}{2 - b} \begin{pmatrix} 2 \\ b \\ 2 \end{pmatrix}.$$

The first column belongs to W exactly when $a = 1$. In all other cases, by Theorem 1.7, the three columns of the matrix are linearly independent and therefore form a basis for $\mathbb{R}^3$ by Corollary 2(b) of Theorem 1.10. So, by part 1, when $b \neq 2$, the matrix is not invertible if and only if $a = 1$.

To summarize the matrix is not invertible precisely when either $b = 2$ or $a = 1$. In other words, the set of pairs $(a, b) \in \mathbb{R}^2$ for which the given matrix is not invertible is the union of two lines, the line $a = 1$ and the line $b = 2$.

Problem D. Suppose $S : \mathbb{R}^3 \to \mathbb{R}^5$ and $T : \mathbb{R}^5 \to \mathbb{R}^3$ are linear transformations that both have rank 3.

(1) Could T have a right inverse? A left inverse? Explain.

Solution — From the April 16 slides, we know that (i) T has a right inverse if and only if it is onto and (ii) T has a left inverse if and only if it is one-to-one.

It is definitely possible for T to be onto. In fact it must be onto, since $\text{rank}(T) = 3 = \dim(\mathbb{R}^3)$. Therefore, by (i), T must have a right inverse.

It is not possible for T to be one-to-one. By the Dimension Theorem, $\text{rank}(T) + \text{nullity}(T) = \dim(\mathbb{R}^5) = 5$. Since $\text{rank}(T) = 3$, we must have $\text{nullity}(T) = 2$. By Theorem 2.4, T is one-to-one exactly when $\mathsf{N}(T) = \{0\}$. Since $\mathsf{N}(T) \neq \{0\}$, T is not one-to-one and hence T does not have a left inverse.

(2) What is the smallest possible rank for TS? Explain and find a pair of such linear transformations where this minimal rank is achieved.

Solution — Since S has rank 3, $\mathsf{R}(S)$ is a 3-dimensional subspace of $\mathbb{R}^5$. Also, $\mathsf{N}(T)$ is a 2-dimensional subspace of $\mathbb{R}^5$ by the Dimension Theorem, since $\text{nullity}(T) = \dim(\mathbb{R}^5) - \text{rank}(T) = 2$ We cannot have $\mathsf{R}(S) \subseteq \mathsf{N}(T)$ since the former has larger dimension than the latter. Therefore TS cannot be the zero transformation, which means that $\text{rank}(TS) \geq 1$ because the zero transformation is the only linear transformation with rank 0.

It turns out that $\text{rank}(TS) = 1$ is achievable. For example, if

$$S(x, y, z) = (x, 0, y, 0, z), \quad T(a_1, a_2, a_3, a_4, a_5) = (a_2, a_3, a_4),$$

both of which have rank 3, then

$$TS(x, y, z) = (0, y, 0)$$

which has rank 1 since $\mathsf{R}(TS) = \text{span}\{(0, 1, 0)\}$.

(3) What is the largest possible rank for ST? Explain and find a pair of such linear transformations where this maximal rank is achieved.

Solution — Since $\mathsf{R}(ST) \subseteq \mathsf{R}(S)$ (because $S(T(x)) \in \mathsf{R}(S)$ for every $x \in \mathbb{R}^5$) we must have $\text{rank}(ST) \leq \text{rank}(S) = 3$. It is possible to have $\text{rank}(ST) = 3$. For example, if

$$S(x, y, z) = (x, y, z, 0, 0), \quad T(a_1, a_2, a_3, a_4, a_5) = (a_1, a_2, a_3),$$

both of which have rank 3, then

$$ST(a_1, a_2, a_3, a_4, a_5) = (a_1, a_2, a_3, 0, 0)$$

has rank 3 too since $\mathsf{R}(ST) = \text{span}\{e_1, e_2, e_3\}$.

In fact, since T is onto, $\mathsf{R}(ST) = \mathsf{R}(S)$ and therefore ST has rank exactly 3.

(4) Could ST be invertible? How about TS? Explain.

Solution — By part 3, we know that $\text{rank}(ST) \leq 3$. Since an invertible linear transformation $\mathbb{R}^5 \to \mathbb{R}^5$ must have rank 5 by the last theorem of the April 16 slides, we see that ST couldn't be invertible.

However, TS can be invertible. This will happen if S is a right inverse of T, as we determined possible in part 1. In fact, as we observed at the end of part 3, TS is always invertible since it always has rank 3.

Problem E. Let V be an n-dimensional vector space over the field F and let I denote the identity transformation on V.

A linear transformation $P : \mathsf{V} \to \mathsf{V}$ is said to have **property Π** if $P^2 = P$, i.e., $P(P(x)) = P(x)$ for every $x \in \mathsf{V}$.

(1) Show that if $\alpha = \{v_1, v_2, \ldots, v_n\}$ is any ordered basis for V and if $0 \le d \le n$, then the linear transformation $P_d^\alpha : \mathsf{V} \to \mathsf{V}$ such that

$$P_d^\alpha(v_i) = \begin{cases} v_i & \text{if } 1 \le i \le d, \\ 0 & \text{if } d+1 \le i \le n, \end{cases}$$

has property Π. (Note that $P_0^\alpha = T_0$ and $P_n^\alpha = I$.)

Solution — Applying the definition of P_d^α twice, we see that

$$P_d^\alpha(P_d^\alpha(v_i)) = \begin{cases} P_d^\alpha(v_i) & \text{if } 1 \le i \le d, \\ P_d^\alpha(0) & \text{if } d+1 \le i \le n, \end{cases} = \begin{cases} v_i & \text{if } 1 \le i \le d, \\ 0 & \text{if } d+1 \le i \le n, \end{cases}.$$

Since α is a basis for V, it follows from the uniqueness part of Theorem 2.6 that $(P_d^\alpha)^2 = P_d^\alpha$.

(2) Show that if $P : \mathsf{V} \to \mathsf{V}$ is a linear transformation with property Π, then $I - P : \mathsf{V} \to \mathsf{V}$ is also a linear transformation with property Π.

Solution — Using the properties listed in Theorem 2.10, we see that

$$(I-P)^2 = (I-P)(I-P) = I(I-P) - P(I-P) = (I-P) - (P-P^2) = I - P,$$

since $P^2 = P$.

(3) Show that if $P : \mathsf{V} \to \mathsf{V}$ is a linear transformation with property Π, then $\mathsf{R}(P) = \mathsf{N}(I-P)$.

Solution — Since $(I-P)P = P - P^2 = P - P = T_0$ by the properties listed in Theorem 2.10, it follows that $\mathsf{R}(P) \subseteq \mathsf{N}(I-P)$. Indeed, if $y \in \mathsf{R}(P)$ then $y = P(x)$ for some $x \in \mathsf{V}$ and then $(I-P)(y) = (I-P)(P(x)) = 0$.

To see that $\mathsf{N}(I-P) \subseteq \mathsf{R}(P)$, suppose $x \in \mathsf{N}(I-P)$. Then $0 = (I-P)(x) = I(x) - P(x) = x - P(x)$. But then $P(x) = x$, which means that $x = P(x) \in \mathsf{R}(P)$.

Since $\mathsf{R}(P) \subseteq \mathsf{N}(I-P)$ and $\mathsf{N}(I-P) \subseteq \mathsf{R}(P)$, we conclude that $\mathsf{R}(P) = \mathsf{N}(I-P)$.

(4) Show that if $P : \mathsf{V} \to \mathsf{V}$ is a linear transformation with property Π, then $\mathsf{N}(P) \cap \mathsf{R}(P) = \{0\}$.

Solution — By part 3, this is equivalent to showing that $\mathsf{N}(P) \cap \mathsf{N}(I - P) = \{0\}$. So suppose $x \in \mathsf{N}(P) \cap \mathsf{N}(I - P)$. Then $0 = P(x)$ and $0 = (I - P)(x) = I(x) - P(x) = x - P(x)$. It follows from this that $x = P(x) = 0$. Since x was an arbitrary element of $\mathsf{N}(P) \cap \mathsf{N}(I - P)$, we conclude that $\mathsf{N}(P) \cap \mathsf{N}(I - P) \subseteq \{0\}$.

Because every subspace of V contains 0, we see that $\mathsf{N}(P) \cap \mathsf{R}(P) = \mathsf{N}(P) \cap \mathsf{N}(I - P) = \{0\}$.

(5) Show that if $P : \mathsf{V} \to \mathsf{V}$ is a linear transformation with property Π, then there are an ordered basis $\alpha = \{v_1, v_2, \dots, v_n\}$ for V and $0 \leq d \leq n$ such that $P = P_d^\alpha$ (as defined in part 1). That is, every linear transformation with property Π is of the form P_d^α described in part 1 for some choice of ordered basis α for V and some choice of $0 \leq d \leq n$.

Solution — Given a linear transformation $P : \mathsf{V} \to \mathsf{V}$ with property Π, we need to find a basis $\alpha = \{v_1, \dots, v_n\}$ and an integer $0 \leq d \leq n$ such that $P = P_d^\alpha$. The number d will be the rank of P. Note that $n - d$ will then be the nullity of P by the Dimension Theorem.

To choose the basis, first pick a basis $\{v_1, \dots, v_d\}$ for $\mathsf{R}(P)$ and then pick a basis $\{v_{d+1}, \dots, v_n\}$ for $\mathsf{N}(P)$ (note that there are appropriately $n - d$ vectors in the latter list). By part 3, $\mathsf{R}(P) \cap \mathsf{N}(P) = \{0\}$ and thus, by Special Assignment 2, $\alpha = \{v_1, \dots, v_d, v_{d+1}, \dots, v_n\}$ forms a basis for the direct sum $\mathsf{R}(P) + \mathsf{N}(P)$. Since V has dimension n and α consists of n linearly independent vectors, it follows from Corollary 2(b) of Theorem 1.10 that α is actually a basis for V (and therefore that $\mathsf{V} = \mathsf{R}(P) + \mathsf{N}(P)$).

I claim that $P = P_d^\alpha$. By Theorem 2.6, it suffices to check that $P(v_i) = P_d^\alpha(v_i)$ for $i = 1, \dots, n$. We consider two cases:

- If $1 \leq i \leq d$, then $v_i \in \mathsf{R}(P) = \mathsf{N}(I - P)$ by part 3. Therefore, $v_i - P(v_i) = 0$ or $P(v_i) = v_i$. By definition of P_d^α, $P_d^\alpha(v_i) = v_i = P(v_i)$, as required.
- If $d + 1 \leq i \leq n$, then $v_i \in \mathsf{N}(P)$, which means that $P(v_i) = 0$. By definition of P_d^α, $P_d^\alpha(v_i) = 0 = P(v_i)$, as required.