Math 24
Spring 2012
Sample Homework Solutions
Week 8

Section 5.2

(2.) Test $A \in M_{2\times 2}(\mathbb{R})$ for diagonalizability, and if possible find an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

(c) $A = \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$. The characteristic polynomial is $(\lambda - 1)(\lambda - 2) - 12 - (\lambda - 5)(\lambda + 2)$, and the roots are 5 and 2, each with multiplicity 1. Because A has two distinct eigenvalues, A is diagonalizable. An eigenvector for $\lambda = 5$ is $(1, 1)$, and an eigenvector for $\lambda = -2$ is $(1, -1)$.
$D = \begin{pmatrix} 5 & 0 \\ 0 & -2 \end{pmatrix}$ $Q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

(d) $A = \begin{pmatrix} 7 & -4 & 0 \\ 8 & -5 & 0 \\ 6 & -6 & 3 \end{pmatrix}$. The characteristic polynomial is $(3 - \lambda)((7 - \lambda)(-5-\lambda)+32) = (3-\lambda)(\lambda-3)(\lambda+1)$, and the roots are 3, with multiplicity 2, and -1, with multiplicity 1. Because $A - 3I$ has rank 1 (and thus nullity 2), A is diagonalizable. Two eigenvectors for $\lambda = 3$ are $(1, 1, 0)$ and $(0, 0, 1)$, and an eigenvector for $\lambda = -1$ is $(2, 4, 3)$.
$D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ $Q = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 4 \\ 0 & 1 & 3 \end{pmatrix}$

(e) $A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$. The characteristic polynomial is $-\lambda((1-\lambda)(-\lambda)+1)+1 = (\lambda^2 + 1)(-\lambda + 1)$, which does not split over $\mathbb{R}$. Therefore A is not diagonalizable.

(3.) Test the operator T on V for diagonalizability, and if T is diagonalizable, find a basis β for V such that $[T]_\beta$ is a diagonal matrix.

(c) $V = \mathbb{R}^3$, and $T(a_1, a_2, a_3) = (a_2, -a_1, 2a_3)$. The matrix of T in the standard basis is $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. The characteristic polynomial is $(2-\lambda)(\lambda^2+1)$, which does not split over $\mathbb{R}$. Therefore T is not diagonalizable.

(d) $V = P_2(\mathbb{R})$ and $T(f(x)) = f(0)+f(1)(x+x^2)$; that is, $T(a+bx+cx^2) = a + (a+b+c)x + (a+b+c)x^2$. The matrix of T in the standard basis is $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. The characteristic polynomial is $(1-\lambda)((1-\lambda)^2-1) = (1-\lambda)(\lambda)(\lambda-2)$, and the roots are 0, 1, and 2. Because there are 3 distinct eigenvalues, T is diagonalizable. A basis of eigenvectors (corresponding to eigenvalues 0, 1, and 2 respectively) is $\{x - x^2,\ 1 - x - x^2,\ x + x^2\}$.

(e) $V = \mathbb{C}^2$, and $T(z, w) = (z + iw, iz + w)$. The matrix of T in the standard basis is $\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$. The characteristic polynomial is $(1-\lambda)^2 + 1$, and the roots are $1+i$ and $1-i$. Because there are 2 distinct eigenvalues, T is diagonalizable. A basis of eigenvectors (corresponding to eigenvalues $1+i$ and $1-i$ respectively) is $\{(1,1),(1,-1)\}$.

(7.) $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$. Find an expression for A^n, where n is an arbitrary positive integer.

Diagonalize A, so $A = QDQ^{-1}$ for a diagonal matrix D. Then $A^n = (QDQ^{-1})^n = QD^nQ^{-1}$. Using the usual methods, we get $D = \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}$ $Q = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}$ $Q^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}$.

$$A^n = \frac{1}{3}\begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 5^n & 0 \\ 0 & (-1)^n \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}.$$

(11.) Let A be an $n \times n$ matrix that is similar to an upper triangular matrix and has distinct eigenvalues $\lambda_1, \ldots, \lambda_k$ with corresponding multiplicities $m_1, \ldots, m_k$. Prove:

(b) $det(A) = (\lambda_1)^{m_1}(\lambda_2)^{m_2}\cdots(\lambda_k)^{m_k}$.

The characteristic polynomial of A is $(\lambda-\lambda_1)^{m_1}(\lambda-\lambda_2)^{m_2}\cdots(\lambda-\lambda_k)^{m_k}$. Let B be the upper triangular matrix similar to A, with diagonal entries $b_1, b_2, \ldots b_n$. Because the determinant of an upper triangular matrix is the product of its diagonal entries, $det(B) = b_1 b_2 \cdots b_n$, and the characteristic polynomial of B is $(\lambda - b_1)(\lambda - b_2)\cdots(\lambda - b_n)$

Because A and B are similar, they have the same characteristic polynomial, so $(\lambda-\lambda_1)^{m_1}(\lambda-\lambda_2)^{m_2}\cdots(\lambda-\lambda_k)^{m_k} = (\lambda - b_1)(\lambda - b_2)\cdots(\lambda - b_n)$. Therefore, $b_1 b_2 \cdots b_n = (\lambda_1)^{m_1}(\lambda_2)^{m_2}\cdots(\lambda_k)^{m_k}$.

Because A and B are similar, they have the same determinant, so $det(A) = det(B) = b_1 b_2 \cdots b_n = (\lambda_1)^{m_1}(\lambda_2)^{m_2}\cdots(\lambda_k)^{m_k}$.

Section 6.1

(3.) In $C([0,1])$, let $f(t) = t$ and $g(t) = e^t$. Compute $\langle f, g\rangle$, $||f||$,$||g||$, and $||f+g||$, and verify the Cauchy-Schwarz inequality ($|\langle x, y\rangle| \leq ||x||\,||y||$) and the triangle inequality ($||x+y|| \leq ||x|| + ||y||$).

$$\langle f, g\rangle = \int_0^1 te^t\,dt = (te^t - e^t)\Big|_0^1 = 1$$

$$||f|| = \sqrt{\int_0^1 t^2\,dt} = \sqrt{\frac{1}{3}}$$

$$||g|| = \sqrt{\int_0^1 e^{2t}\,dt} = \sqrt{\frac{e^{2t}}{2}\Big|_0^1} = \sqrt{\frac{e^2-1}{2}}$$

$$||f+g|| = \sqrt{\int_0^1 (t+e^t)^2\,dt} = \sqrt{\int_0^1 t^2 + 2te^t + e^{2t}\,dt} = \sqrt{\frac{1}{3} + 2 + \frac{e^2-1}{2}}$$

To verify the Cauchy-Schwarz inequality, we see $|\langle f, g\rangle| = 1$, and $||f||\,||g|| = \sqrt{\dfrac{e^2-1}{6}}$. Since $e^2 - 1 > 6$, we have $|\langle f, g\rangle| \leq ||f||\,||g||$.

To verify the triangle inequality, since all quantities are non-negative, we can check that $(||f+g||)^2 \leq (||f|| + ||g||)^2$. We have $(||f+g||)^2 = \frac{1}{3} + 2 + \frac{e^2-1}{2}$ and $(||f|| + ||g||)^2 = \frac{1}{3} + 2\sqrt{\frac{1}{3}}\sqrt{\frac{e^2-1}{2}} + \frac{e^2-1}{2}$, so we need to check that $\sqrt{\frac{1}{3}}\sqrt{\frac{e^2-1}{2}} > 1$. Since $e^2 - 1 > 6$, this is true.

(10.) Let V be an inner product space,and suppose that x and y are orthogonal vectors in V. Prove that $||x+y||^2 = ||x||^2 + ||y||^2$. Deduce the Pythagorean Theorem in $\mathbb{R}^2$.

Since x and y are orthogonal, $\langle x,\, y\rangle = \langle y,\, x\rangle = 0$. Therefore,

$$||x+y||^2 = \langle x+y,\, x+y\rangle = \langle x,\, x+y\rangle + \langle y,\, x+y\rangle =$$

$$\langle x,\, x\rangle + \langle x,\, y\rangle + \langle y,\, x\rangle + \langle y,\, y\rangle = \langle x,\, x\rangle + \langle y,\, y\rangle = ||x||^2 + ||y||^2.$$

In $\mathbb{R}^2$, letting x and y denote the legs of a right triangle (both emanating from the right angle), so the lengths of the legs are $a = ||x||$ and $b = ||y||$, the hypoteneuse is $x - y$ and the length of the hypoteneuse is $c = ||x - y||$. Since x and y are orthogonal, so are x and $-y$. Therefore by the theorem, $|||x||^2 + ||y||^2 = ||x-y||^2$, or $a^2 + b^2 = c^2$.

(17.) Let T be a linear operator on an inner product space V, and suppose that $||T(x)|| = ||x||$ for all x. Prove that T is one-to-one.

If $x \neq 0$, then $||x|| > 0$, so $||T(x)|| > 0$, and $T(x) \neq 0$. Therefore, $N(T) = \{0\}$, and T is one-to-one.

Additional problem from Wednesday:

Suppose V is an n-dimensional vector field over the field F, where F is either $\mathbb{R}$ or $\mathbb{C}$, and $\langle , \rangle$ denotes the standard inner product on F^n. Let $\beta = \{v_1, v_2, \ldots, v_n\}$ be an ordered basis for V. For $v, w \in V$, define

$$\langle\langle v, w\rangle\rangle = \langle [v]_\beta, [w]_\beta\rangle\,.$$

(a.) Show that $\langle\langle , \rangle\rangle$ is an inner product on V.
(b.) Show that β is an orthonormal set for this inner product.

To show that $\langle\langle , \rangle\rangle$ satisfies the definition of an inner product, we use the fact that the function taking v to $[v]_\beta$ is an isomorphism, along with the fact that $\langle , \rangle$ is an inner product on F^n.

$$\langle\langle x+y,\, z\rangle\rangle = \langle [x+y]_\beta,\, [z]_\beta\rangle = \langle [x]_\beta + [y]_\beta,\, [z]_\beta\rangle =$$

$$\langle [x]_\beta,\, [z]_\beta\rangle + \langle [y]_\beta,\, [z]_\beta\rangle = \langle\langle x,\, z\rangle\rangle + \langle\langle y,\, z\rangle\rangle$$

$$\langle\langle cx,\, z\rangle\rangle = \langle [cx]_\beta,\, [z]_\beta\rangle = \langle c[x]_\beta,\, [z]_\beta\rangle = c\,\langle [x]_\beta,\, [z]_\beta\rangle = c\,\langle\langle x,\, z\rangle\rangle$$

$$\langle\langle y,\, x\rangle\rangle = \langle [y]_\beta,\, [x]_\beta\rangle = \overline{\langle [x]_\beta,\, [y]_\beta\rangle} = \overline{\langle\langle x,\, y\rangle\rangle}$$

If $x \neq 0$ then $[x]_\beta \neq 0$, and so

$$\langle\langle x,\, x\rangle\rangle = \langle [x]_\beta,\, [x]_\beta\rangle > 0.$$

To show that β is an orthonormal set, we use the fact that $[v_i]_\beta = e_i$. Therefore

$$\langle\langle v_i,\, v_j\rangle\rangle = \langle e_i,\, e_j\rangle = \begin{cases} 1 & i = j; \\ 0 & i \neq j. \end{cases}$$

Section 6.2

(2.) Apply the Gram-Schmidt process to S to obtain an orthogonal basis for $span(S)$. Normalize the vectors to obtain an orthonormal basis β. Compute the Fourier coefficients of the given vector relative to β. Use Theorem 6.5 to check your result.

Theorem 6.5 says that if $\beta = \{v_1, v_2, \ldots, v_n\}$ and $vinspan(S)$, then $x = a_1v_1 + a_2v_2 + \cdots + a_nv_n$, where $a_1, a_2, \ldots, a_n$ are the Fourier coefficients of v relative to β.

(a.) $V = \mathbb{R}^3$, $S = \{(1,1,1),\ (0,1,1),\ (1,3,3)\}$ and $x = (1,1,2)$.

$v_1 = (1,0,1)$.
$v_2 = (0,1,1) - \frac{\langle (0,1,1),\,(1,0,1)\rangle}{\langle (1,0,1),\,(1,0,1)\rangle}(1,0,1) = (0,1,1) - \frac{1}{2}(1,0,1) = (-\frac{1}{2},\ 1,\ \frac{1}{2})$.
$v_3 = (1,3,3) - \frac{\langle (1,3,3),\,(1,0,1)\rangle}{\langle (1,0,1),\,(1,0,1)\rangle}(1,0,1) - \frac{\left\langle (1,3,3),\,(-\frac{1}{2},1,\frac{1}{2})\right\rangle}{\left\langle (-\frac{1}{2},1,\frac{1}{2}),\,(-\frac{1}{2},1,\frac{1}{2})\right\rangle}(-\frac{1}{2},\ 1,\ \frac{1}{2}) =$
$(1,3,3) - \frac{4}{2}(1,0,1) - \frac{4}{\frac{3}{2}}(-\frac{1}{2},\ 1,\ \frac{1}{2}) = (\frac{1}{3},\ \frac{1}{3},\ -\frac{1}{3})$.
Normalizing these vectors:
$\beta = \{(\frac{\sqrt{2}}{2},\ 0,\ \frac{\sqrt{2}}{2}),\ (-\frac{\sqrt{6}}{6},\ \frac{2\sqrt{6}}{6},\ \frac{\sqrt{6}}{6}),\ (\frac{\sqrt{3}}{3},\ \frac{\sqrt{3}}{3},\ -\frac{\sqrt{3}}{3})\}$.
The Fourier coefficients of $(1,1,2)$ are:
$\left\langle (1,1,2),\ (\frac{\sqrt{2}}{2},\ 0,\ \frac{\sqrt{2}}{2})\right\rangle = \frac{3\sqrt{2}}{2}$
$\left\langle (1,1,2),\ (-\frac{\sqrt{6}}{6},\ \frac{2\sqrt{6}}{6},\ \frac{\sqrt{6}}{6})\right\rangle = \frac{\sqrt{6}}{2}$
$\left\langle (1,1,2),\ (\frac{\sqrt{3}}{3},\ \frac{\sqrt{3}}{3},\ -\frac{\sqrt{3}}{3})\right\rangle = 0.$
To check:
$\frac{3\sqrt{2}}{2}(\frac{\sqrt{2}}{2},\ 0,\ \frac{\sqrt{2}}{2}) + \frac{\sqrt{6}}{2}(-\frac{\sqrt{6}}{6},\ \frac{2\sqrt{6}}{6},\ \frac{\sqrt{6}}{6}) + 0(\frac{\sqrt{3}}{3},\ \frac{\sqrt{3}}{3},\ -\frac{\sqrt{3}}{3}) = (\frac{3}{2},\ 0,\ \frac{3}{2}) + (-\frac{1}{2},\ 1,\ \frac{1}{2}) +$
$(0,0,0) = (1,1,2)$.

(c.) $V = P_2(\mathbb{R})$, $\langle f(x),\, g(x)\rangle = \int_0^1 f(x)g(x)\,dx$, $X = \{1, x, x^2\}$, $h(x) = 1 + x$.

$v_1 = 1$
$v_2 = x - \frac{\langle x,1\rangle}{\langle 1,1\rangle}1 = x - \frac{\int_0^1 x\,dx}{\int_0^1 1\,dx} = x - \frac{1}{2}$
$v_3 = x^2 - \frac{\left\langle x^2,1\right\rangle}{\langle 1,1\rangle}1 - \frac{\left\langle x^2, x-\frac{1}{2}\right\rangle}{\left\langle x-\frac{1}{2}, x-\frac{1}{2}\right\rangle}(x - \frac{1}{2}) = x^2 - \frac{\int_0^1 x^2\,dx}{\int_0^1 1\,dx} - \frac{\int_0^1 x^3 - \frac{1}{2}x^2\,dx}{\int_0^1 x^2 - x + \frac{1}{4}\,dx}(x - \frac{1}{2}) = x^2 - \frac{1}{3} - (1)(x - \frac{1}{2}) = x^2 - x + \frac{1}{6}$
Normalizing these vectors:
$\beta = \{1,\, 2\sqrt{3}x - \sqrt{3},\, \frac{6\sqrt{55}}{11}x^2 - \frac{6\sqrt{55}}{11}x + \frac{\sqrt{55}}{11}\}$.
The Fourier coefficients of $1 + x$ are:
$\langle 1 + x,\, 1\rangle = \int_0^1 (1 + x)(1)\,dx = \frac{3}{2}$
$\left\langle 1 + x,\, 2\sqrt{3}x - \sqrt{3}\right\rangle = \int_0^1 (1 + x)(2\sqrt{3}x - \sqrt{3})\,dx = \frac{\sqrt{3}}{6}$
$\left\langle 1 + x,\, \frac{6\sqrt{55}}{11}x^2 - \frac{6\sqrt{55}}{11}x + \frac{\sqrt{55}}{11}\right\rangle = \int_0^1 (1 + x)(\frac{6\sqrt{55}}{11}x^2 - \frac{6\sqrt{55}}{11}x + \frac{\sqrt{55}}{11})\,dx = 0$
To check:
$\frac{3}{2}(1) + \frac{\sqrt{3}}{6}(2\sqrt{3}x - \sqrt{3}) + 0(\frac{6\sqrt{55}}{11}x^2 - \frac{6\sqrt{55}}{11}x + \frac{\sqrt{55}}{11}) = x + 1.$

(d.) $V = span(S)$, $S = \{(1, i, 0),\, (1 - i,\, 2,\, 4i)\}$, $x = (3 + i,\, 4i, -4)$.

$v_1 = (1, i, 0)$
$v_2 = (1-i,\, 2,\, 4i) - \frac{\langle (1-i, 2, 4i),\, (1,i,0)\rangle}{\langle (1,i,0),\, (1,i,0)\rangle}(1, i, 0) = (1-i,\, 2,\, 4i) - \frac{(1-i)(1)+(2)(-i)+(4i)(0)}{(1)(1)+(i)(-i)+(0)(0)}(1, i, 0) = (1 - i,\, 2,\, 4i) - \frac{1-3i}{2}(1, i, 0) = (1 - i,\, 2,\, -4i) - (\frac{1-3i}{2},\, \frac{3+i}{2},\, 0) = (\frac{1+i}{2},\, \frac{1-i}{2},\, 4i)$
Normalizing these vectors:
$\beta = \{\frac{\sqrt{2}}{2}(1,\, i,\, 0),\, \frac{\sqrt{17}}{34}(1 + i,\, 1 - i,\, 8i)\}$.
The Fourier coefficients of $(3 + i,\, 4i,\, -4)$ are:
$\left\langle (3 + i,\, 4i,\, -4),\, \frac{\sqrt{2}}{2}(1,\, i,\, 0)\right\rangle = \frac{\sqrt{2}}{2}((3 + i)(1) + (4i)(-i) + (-4)(0) = \frac{\sqrt{2}}{2}(7 + i)$
$\left\langle (3 + i,\, 4i,\, -4),\, \frac{\sqrt{17}}{34}(1 + i,\, 1 - i,\, 8i)\right\rangle = \frac{\sqrt{17}}{34}((3 + i)(1 - i) + (4i)(1 + i) + (-4)(-8i) = \frac{\sqrt{17}}{34}(34i)$
To check:
$\frac{\sqrt{2}}{2}(7 + i)\frac{\sqrt{2}}{2}(1,\, i,\, 0) + \frac{\sqrt{17}}{34}(34i)\frac{\sqrt{17}}{34}(1 + i,\, 1 - i,\, 8i) = (3 + i, 4i, -4)$

(7.) Let β be a basis for a subspace W of an inner product space V, and let $z \in V$. Prove that $z \in W^{\perp}$ if and only if $\langle z, v\rangle = 0$ for every $z \in \beta$.

By definition, if $z \in W^{\perp}$ then z is orthogonal to every $v \in W$, so in particular, $\langle z, v\rangle = 0$ for every $v \in \beta$.

For the converse, suppose that $\langle z, v\rangle = 0$ for every $v \in \beta$. To show $z \in W^{\perp}$, let w be any element of W. We must show $\langle z, w\rangle = 0$.

Since β is a basis for W, we can write w as a linear combination of elements of β, as $w = a_1v_1+\cdots+a_kv_k$. Now $\langle z, w\rangle = \langle z, a_1v_1 + \cdots + a_kv_k\rangle = \langle z, a_1v_1\rangle+\cdots+\langle z, a_kv_k\rangle = \overline{a_1}\langle z, v_1\rangle+\cdots+\overline{a_k}\langle z, v_k\rangle = \overline{a_1}(0)+\cdots+\overline{a_k}(0) = 0$.

Notice here: We used only that β generates W, not that β is linearly independent. Therefore, we have shown that $\beta^{\perp} = (span(\beta))^{\perp}$ for any $\beta \subseteq V$.

(9.) Let $W = span(\{(i, 0, 1)\})$ in $\mathbb{C}^3$. Find orthonormal bases for W and $W^{\perp}$.

An orthonormal basis for W is $\frac{\sqrt{2}}{2}(i, 0, 1)$. An orthonormal basis for $W^{\perp}$ is $\{\frac{\sqrt{2}}{2}(1, 0, i), (0, 1, 0)\}$.

(12.) Prove for every matrix $A \in M_{m\times n}(F)$, $(R(L_{A^*}))^{\perp} = N(L_A)$.

For the purposes of this proof, let $\cdot$ denote the "dot product", so that $(x_1, x_2, \ldots, x_n) \cdot (y_1, y_2, \ldots, y_n) = x_1y_1 + x_2y_2 + \cdots + x_ny_n$. Thus, if the rows of A are $r_1, r_2, \ldots, r_m \in F^m$, and $x \in F^n$, then the entries of Ax are $x \cdot r_1, x \cdot r_2, \ldots, x \cdot r_m$.

Also, if $y = (y_1, y_2, \ldots, y_n)$, let $\overline{y}$ denote $(\overline{y_1}, \overline{y_2}, \ldots, \overline{y_n})$. Notice that we can define the standard inner product by $\langle x, y\rangle = x \cdot \overline{y}$. This is equivalent to $\langle x, \overline{y}\rangle = x \cdot y$.

Now $x \in N(L_A)$ if and only if all the entries of Ax are zero; that is, if and only if $x \cdot r_i = 0$ for $i = 1, 2, \ldots, m$; or, if and only if $\langle x, \overline{r_i}\rangle = 0$ for $i = 1, 2, \ldots, m$. Now $\overline{r}_i$ is the i^{th} column of A^*. Therefore, we have shown that $x \in N(L_A)$ if and only if x is orthogonal to every column of A^*. Let S be the set of columns of A^*; then $x \in N(L_A)$ if and only if $x \in S^{\perp}$, so $N(L_A) = S^{\perp}$.

The span of S is $R(L_{A^*})$. If we show $S^{\perp} = (span(S))^{\perp}$, we will be done. But that is just what we showed in problem (7).