Math 24
Spring 2012
Sample Homework Solutions
Week 7

Section 4.3

(3.) Use Cramer's Rule to solve the system of linear equations

$$2x_1 + x_2 - 3x_3 = 5$$

$$x_1 - 2x_2 + x_3 = 10$$

$$3x_1 + 4x_2 - 2x_3 = 0$$

In each case, I first simplify the determinant, in the first case by subtracting row 1 and row 2 from row 3, and in the other cases, by subtracting 2 times row 1 from row 2.

$$\begin{vmatrix} 2 & 1 & -3 \\ 1 & -2 & 1 \\ 3 & 4 & -2 \end{vmatrix} = \begin{vmatrix} 2 & 1 & -3 \\ 1 & -2 & 1 \\ 0 & 5 & 0 \end{vmatrix} = -5 \begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} = -25$$

$$\begin{vmatrix} 5 & 1 & -3 \\ 10 & -2 & 1 \\ 0 & 4 & -2 \end{vmatrix} = \begin{vmatrix} 5 & 1 & -3 \\ 0 & -4 & 7 \\ 0 & 4 & -2 \end{vmatrix} = 5 \begin{vmatrix} -4 & 7 \\ 4 & -2 \end{vmatrix} = -100$$

$$\begin{vmatrix} 2 & 5 & -3 \\ 1 & 10 & 1 \\ 3 & 0 & -2 \end{vmatrix} = \begin{vmatrix} 2 & 5 & -3 \\ -3 & 0 & 7 \\ 3 & 0 & -2 \end{vmatrix} = 5 \begin{vmatrix} -3 & 7 \\ 3 & -2 \end{vmatrix} = 75$$

$$\begin{vmatrix} 2 & 1 & 5 \\ 1 & -2 & 10 \\ 3 & 4 & 0 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 5 \\ -3 & -4 & 0 \\ 3 & 4 & 0 \end{vmatrix} = 0$$

$$x_1 = \frac{-100}{-25} = 4 \qquad x_2 = \frac{75}{-25} = -3 \qquad x_3 = \frac{0}{-25} = 0$$

(12.) A matrix Q is called orthogonal if $QQ^t = I$. Prove that if Q is orthogonal then $det(Q) = \pm 1$.

We know that $det(Q^t) = det(Q)$, and that $det(AB) = det(A)det(B)$. Therefore, if Q is orthogonal, we have

$$1 = det(I) = det(QQ^t) = det(Q)det(Q^t) = det(Q)det(Q) = (det(Q))^2,$$

and therefore $det(Q) = \pm 1$.

Section 4.4

(4.) Evaluate the determinant by any legitimate method

(e) $\begin{pmatrix} i & 2 & -1 \\ 3 & i+1 & 2 \\ -2i & 1 & 4-i \end{pmatrix}$

Simplify by adding multiples of row 1 to rows 2 and 3:

$$\begin{vmatrix} i & 2 & -1 \\ 3 & i+1 & 2 \\ -2i & 1 & 4-i \end{vmatrix} = \begin{vmatrix} i & 2 & -1 \\ 0 & 1+7i & 2-3i \\ 0 & 5 & 2-i \end{vmatrix} = i \begin{vmatrix} 1+7i & 2-3i \\ 5 & 2-i \end{vmatrix} = -28 - i$$

(g) $\begin{pmatrix} 1 & 0 & -2 & 3 \\ -3 & 1 & 1 & 2 \\ 0 & 4 & -1 & 1 \\ 2 & 3 & 0 & 1 \end{pmatrix}$

Simplify by adding multiples of row 1 to rows 2 and 4:

$$\begin{vmatrix} 1 & 0 & -2 & 3 \\ -3 & 1 & 1 & 2 \\ 0 & 4 & -1 & 1 \\ 2 & 3 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & -5 & 11 \\ 0 & 4 & -1 & 1 \\ 0 & 3 & 4 & -5 \end{vmatrix} = \begin{vmatrix} 1 & -5 & 11 \\ 4 & -1 & 1 \\ 3 & 4 & -5 \end{vmatrix}$$

Simplify by adding multiples of row 1 to rows 2 and 3:

$$\begin{vmatrix} 1 & -5 & 11 \\ 4 & -1 & 1 \\ 3 & 4 & -5 \end{vmatrix} = \begin{vmatrix} 1 & -5 & 11 \\ 0 & 19 & -43 \\ 0 & 19 & -38 \end{vmatrix} = \begin{vmatrix} 19 & -43 \\ 19 & -38 \end{vmatrix} = 19 \begin{vmatrix} 1 & -43 \\ 1 & -38 \end{vmatrix} = 95$$

Section 3.2

(21.) Let A be an $m \times n$ matrix with rank m. Prove that there exists an $n \times m$ matrix B such that $AB = I_m$.

We know that $AB = I_m$ if and only if $L_{AB} = I_{F^m}$; that is, if and only if $L_A L_B = I_{F^m}$. So we need to find a linear transformation $T : F_m \to F_n$ such that $L_A T = I_{F^m}$, and then let B be the matrix of T relative to the standard bases, so $T = L_B$.

We know that $L_A : F^n \to F^m$ has rank m; that is, it is onto.

We need to find $T : F^m \to F^n$ such that $L_A(T(v)) = v$ for all $v \in F^m$. We use the fact that we can define a linear transformation however we like on the elements of a basis.

Because L_A is onto, we can choose $v_i \in F^n$ such that $L_A(v_i) = e_i$ for $i = 1, \ldots, m$. Then define T so that $T(e_i) = v_i$ for $i = 1, \ldots, m$. Therefore, on the standard basis for F^m, we have $L_A(T(e_i)) = L_A(v_i) = e_i = I_{F^m}(e_i)$. Now, because $L_A T$ equals I_{F^n} on the standard basis, and linear transformations are determined by their action on a basis, we have $L_A T = I_{F^n}$, which is what we needed.

We might note that the i^{th} column of B is the vector v_i we chose such that $L_A(v_i) = e_i$, that is, $Av_i = e_i$.

If $n > m$, then L_A is onto but not one-to-one, so there are many possible choices for v_i, and therefore many possible choices for B.

Section 4.3

(21.) Prove that if $M \in M_{n \times n}(F)$ can be written in the form $M = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$, where A and C are square matrices, then $det(M) = det(A)det(C)$.

Suppose that A is $m \times m$. By type 3 elementary row operations using rows 1 through m of M, convert A to upper triangular form so $M = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$

becomes $M^* = \begin{pmatrix} A^* & B^* \\ 0 & C \end{pmatrix}$. Now, by type 3 elementary row operations using rows $m+1$ through n of M^*, convert C to upper triangular form so $M^* = \begin{pmatrix} A^* & B^* \\ 0 & C \end{pmatrix}$ becomes $M^{**} = \begin{pmatrix} A^* & B^* \\ 0 & C^* \end{pmatrix}$.

Because type 3 elementary operations do not change the determinant, $det(M^{**}) = det(M)$, $det(A^*) = det(A)$, and $det(C^*) = det(C)$. But because M^{**}, A^*, and C^* are upper triangular, their determinants are the products of their diagonal entries, so

$$det(M) = det(M^{**}) = det(A^*)det(C^*) = det(A)det(C).$$

Section 5.1

(3.) Find the eigenvalues, corresponding eigenvectors, and, if possible, a basis of eigenvectors and an invertible Q and diagonal B such that $Q^{-1}AQ = D$.

The eigenvalues of A are the roots of the characteristic polynomial, $A-\lambda I$. For a given λ, the eigenvectors are the nonzero elements of the null space of $A - \lambda I$, which we can find by row-reducing $A - \lambda I$.

If there is a basis β of eigenvectors, Q will be the matrix that changes from β-coordinates to standard coordinates (its columns will be the vectors in β), and the diagonal entries of D will be the eigenvalues corresponding to the eigenvectors of β.

(c) $A = \begin{pmatrix} i & 1 \\ 2 & -i \end{pmatrix}$; $F = \mathbb{C}$.

The characteristic polynomial is $(i-\lambda)(-i-\lambda)-2 = \lambda^2-1$; the eigenvalues are $\lambda = 1$ and $\lambda = -1$, and corresponding eigenvectors are $t(1, 1-i)$ $(t \neq 0)$ and $t(1, -1-i)$ $(t \neq 0)$. A basis of eigenvectors is $\{(1, 1-i), (1, -1-i)\}$, $Q = \begin{pmatrix} 1 & 1 \\ 1-i & -1-i \end{pmatrix}$, $D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

(d) $A = \begin{pmatrix} 2 & 0 & -1 \\ 4 & 1 & -4 \\ 2 & 0 & -1 \end{pmatrix}$; $F = \mathbb{R}$.

The characteristic polynomial is $(1-\lambda)((2-\lambda)(-1-\lambda)+2) = -\lambda(1-\lambda)^2$; the eigenvalues are $\lambda = 0$, of multiplicity 1, and $\lambda = 1$, of multiplicity 2, and corresponding eigenvectors are $t(1,4,2)$ $(t \neq 0)$ and $s(1,0,1)+t(0,1,0)$ $((s,t) \neq (0,0))$. A basis of eigenvectors is $\{(1,4,2),(1,0,1),(0,1,0)\}$, $Q = \begin{pmatrix} 1 & 1 & 0 \\ 4 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}$, $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

(4.) Find the eigenvalues and an ordered basis β such that $[T]_\beta$ is a diagonal matrix.

If the vector space is not F^n, start with any basis α. The eigenvalues are the eigenvalues of $[T]_\alpha$, and the eigenvectors are the vectors v such that $[v]_\alpha$ is an eigenvector of $[T]_\alpha$. Then β will consist of eigenvectors. (That is, solve the problem on the coordinate level, then move your solution back.)

(a) $V = \mathbb{R}^2$ and $T(a,b) = (-2a+3b, -10a+9b)$.

Let α be the standard basis for $\mathbb{R}^2$. Then the matrix $[T]_\alpha$ is $\begin{pmatrix} -2 & 3 \\ -10 & 9 \end{pmatrix}$, which has eigenvalues 3 and 4 with corresponding eigenvectors $(3,5)$ and $(1,2)$. Therefore T has eigenvalues 3 and 4, and a basis of eigenvectors is $\{(3,5),(1,2)\}$.

(d) $V = P_1(\mathbb{R})$ and $T(ax+b) = (-6a+2b)x + (-6a+b)$.

Let $\alpha = \{x, 1\}$ be a basis for $P_1(\mathbb{R})$. (Note, this is not the standard ordered basis, which is $\{1, x\}$.) Then the matrix $[T]_\alpha$ is $\begin{pmatrix} -6 & 2 \\ -6 & 1 \end{pmatrix}$, which has eigenvalues -3 and -2 with corresponding eigenvectors $(2,3)$ and $(1,2)$. Therefore T has eigenvalues -3 and -2, and a basis of eigenvectors is $\{2x+3, x+2\}$.

(h) $V = M_{2\times 2}(\mathbb{R})$ and $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & b \\ c & a \end{pmatrix}$.

Let α be the standard basis for $M_{2\times 2}(\mathbb{R})$. Then the matrix $[T]_\alpha$ is $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 9 & 0 & 0 & 1 \end{pmatrix}$, which has eigenvalues 1, of multiplicity 3, and -1, of multiplicity 1, with corresponding eigenvectors $(1,0,0,0)$, $(0,1,1,0)$, $(0,0,0,1)$ for 1 and $(0,1,-1,0)$ for -1.

Therefore T has eigenvalues 1, of multiplicity 3, and -1, of multiplicity 1, and a basis of eigenvectors is $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$.

(7a) Let T be a linear operator on a finite-dimensional vector space V. We define the *determinant* of T, denoted $det(T)$, as follows: Choose any ordered basis β for V, and define $det(T) = det([T]_\beta)$.

Prove that this definition is independent of the choice of an ordered basis for V. That is, prove that if β and γ are two ordered bases for V, then $det([T]_\beta) = det([T]_\gamma)$.

We know, from our special homework assignments, that we are proving that the determinant of T is *well-defined*.

Since $[T]_\beta$ and $[T]_\gamma$ are similar matrices, by property (9) on page 236 they have the same determinant: $det([T]_\beta) = det([T]_\gamma)$.