Math 24
Spring 2012
Sample Homework Solutions
Week 6

Section 3.2

(19.) Let A be an $m \times n$ matrix with rank m and B be an $n \times p$ matrix with rank n. Determine the rank of AB. Justify your answer.

By the correspondence between matrices and linear transformations, we can rephrase this problem as follows: $L_A : F^n \to F^m$ has rank m, and $L_B : F^p \to F^n$ has rank n. Find the rank of $L_{AB} = L_A L_B : F^p \to F^m$.

Since L_B has rank n, its range has dimension n; since its codomain F^n has dimension n, this means L_B must be onto; its range is F^n, the entire domain of L_A. Therefore anything in the range of L_A is in the range of $L_A L_B$, so $L_A L_B$ has the same rank as L_A. Therefore L_{AB} has rank m, as does AB.

Notice the general result: If T is onto, then U and UT have the same range, hence the same rank. You showed the "dual" fact in problem 5(b) of the take-home midterm: If U is one-to-one, then T and UT have the same null space, hence the same nullity.

Section 3.3

(1.) The answers are in the back of the book.

(2d.) Find the dimension of, and a basis for, the solution set.

$$2x_1 + x_2 - x_3 = 0$$

$$x_1 - x_2 + x_3 = 0$$

$$x_1 + 2x_2 - 2x_3 = 0$$

This system is converted by row reduction into the following system.

$$x = 0$$

$$y - z = 0$$

We can see that the general solution is $x = 0$, $y = z$, or $(x, y, z) = t(0, 1, 1)$; the solution set has dimension 1 and a basis is $\{(0, 1, 1)\}$.

(3d.) Find the general solution, given that one solution is $(x, y, z) = (2, 2, 1)$.

$$2x_1 + x_2 - x_3 = 5$$

$$x_1 - x_2 + x_3 = 1$$

$$x_1 + 2x_2 - 2x_3 = 4$$

The corresponding homogeneous system, by problem (2d) has solution $(x, y, z) = t(0, 1, 1)$, so this system has solution $(x, y, z) = t(0, 1, 1) + (2, 2, 1)$.

(7c.) Use Theorem 3.1 to determine whether the system has a solution:

$$x_1 + 2x_2 + 3x_3 = 1$$

$$x_1 + x_2 - x_3 = 0$$

$$x_1 + 2x_2 + x_3 = 3$$

The coefficient matrix of this system is $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & -1 \\ 1 & 2 & 1 \end{pmatrix}$, and the augmented matrix is $\begin{pmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & -1 & 0 \\ 1 & 2 & 1 & 3 \end{pmatrix}$. Since both matrices have three linearly independent rows, they both have rank 3, and since they have the same rank, the system has a solution.

Section 3.4

(3.) Suppose that the augmented matrix of a system $Ax = b$ is transformed into a matrix $(A'|b')$ in reduced row echelon form by a finite sequence of elementary row operations.

(a.) Prove that $rank(A') \neq rank(A'|b')$ if and only if $(A'|b')$ contains a row in which the only nonzero entry lies in the last column.

Since $(A'|b')$ is in reduced row echelon form, all its nonzero rows are linearly independent, and the same applies to A'. Therefore, $rank(A') \neq rank(A'|b')$ if and only if, for some i, row i is a zero row of A' but a nonzero row of $(A'|b')$. This can happen if and only if the only nonzero entry of row i is in the last column.

(b.) Deduce that $Ax = b$ is consistent if and only if $(A'|b')$ contains no row in which the only nonzero entry is in the last column.

By Theorem 3.1 $Ax = b$ is consistent if and only if $rank(A) = rank(A|b)$. Since elementary row operations do not change the rank of a matrix, this is true if and only if $rank(A') = rank(A'|b')$. By part (a), this is true if and only if $(A'|b')$ contains no row in which the only nonzero entry lies in the last column.

(4.) For each system, apply (3) to determine whether it is consistent; if it is, find all solutions; find a basis for the solution set of the corresponding homogeneous system.

(a.)

$$x_1 + 2x_2 - x_3 + x_4 = 2$$
$$2x_1 + x_2 + x_3 - x_4 = 3$$
$$x_1 + 2x_2 - 3x_3 + 2x_4 = 2$$

The reduced row echelon form of the augmented matrix of this system is $\begin{pmatrix} 1 & 0 & 0 & -\frac{1}{2} & \frac{4}{3} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{2} & 0 \end{pmatrix}$. By (3), the system is consistent. The general solution is $\left(\frac{4}{3}, \frac{1}{3}, 0, 0\right) + t\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, 1\right)$, and a basis for the solution set of the corresponding homogeneous system is $\left\{\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, 1\right)\right\}$.

(b.)

$$x_1 + x_2 - 3x_3 + x_4 = -2$$
$$x_1 + x_2 + x_3 - x_4 = 2$$
$$x_1 + x_2 - x_3 = 0$$

The reduced row echelon form of the augmented matrix of this system is $\begin{pmatrix} 1 & 1 & 0 & -\frac{1}{2} & 1 \\ 0 & 0 & 1 & -\frac{1}{2} & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. By (3), the system is consistent. The general solution is $(1,0,1,0)+s(-1,1,0,0)+t\left(\frac{1}{2},\ 0,\ \frac{1}{2},\ 1\right)$, and a basis for the solution set of the corresponding homogeneous system is $\left\{(-1,1,0,0),\ \left(\frac{1}{2},\ 0,\ \frac{1}{2},\ 1\right)\right\}$.

The reduced row echelon form of the augmented matrix of this system is The reduced row echelon form of the augmented matrix of this system is $\begin{pmatrix} 1 & 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$. By (3), the system is inconsistent. The coefficient matrix is the same as in (b), so a basis for the solution set of the corresponding homogeneous system is $\left\{(-1,1,0,0),\ \left(\frac{1}{2},\ 0,\ \frac{1}{2},\ 1\right)\right\}$.

(7.) It can be shown that the vectors $u_1 = (2,-3,1)$, $u_2 = (1,4,-2)$, $u_3 = (-8,12,-4)$, $u_4 = (1,37,-17)$, and $u_5 = (-3,-5,8)$ genreate $\mathbb{R}^3$. Find a subset of $\{u_1,u_2,u_3,u_4,u_5\}$ that is a basis for $\mathbb{R}^3$.

In the reduced row echelon form of the matrix whose columns are u_1, u_2, u_3, u_4, and u_5, columns 1, 2 and 5 are linearly independent. Therefore, $\{u_1,u_2,u_5\}$ is linearly independent, and is a basis for $\mathbb{R}^3$.

Section 4.1

(3a.) Find $det\begin{pmatrix} -1+i & 1-4i \\ 3+2i & 2-3i \end{pmatrix}$.

$(-1+i)(2-3i)-(3+2i)(1-4i) = -10+15i$.

Section 4.2

(10.) Find $det\begin{pmatrix} i & 2+i & 0 \\ -1 & 3 & 2i \\ 0 & -1 & 1-i \end{pmatrix}$ by cofactor expansion along the second row.

$$-(-1)\begin{vmatrix} 2+i & 0 \\ -1 & 1-i \end{vmatrix} + 3\begin{vmatrix} i & 0 \\ 0 & 1-i \end{vmatrix} - 2i\begin{vmatrix} i & 2+i \\ 0 & -1 \end{vmatrix} = 4+2i.$$

(22.) Find $\begin{vmatrix} 1 & -2 & 3 & -12 \\ -5 & 12 & -14 & 19 \\ -9 & 22 & -10 & 31 \\ -4 & 9 & -14 & 14 \end{vmatrix}$.

We use the fact that type 3 row and column operations do not change the determinant. Adding multiples of row 1 to the other rows gives
$\begin{vmatrix} 1 & -2 & 3 & -12 \\ 0 & 22 & -19 & 79 \\ 0 & 4 & 17 & -77 \\ 0 & 1 & -2 & -34 \end{vmatrix}$, and cofactor expansion along the first column gives
$\begin{vmatrix} 22 & -19 & 79 \\ 4 & 17 & -77 \\ 1 & -2 & -34 \end{vmatrix}$. Adding multiples of the third row to the other two rows gives $\begin{vmatrix} 0 & 25 & 827 \\ 0 & 25 & 59 \\ 1 & -2 & -34 \end{vmatrix}$, and cofactor expansion along the first column gives
$\begin{vmatrix} 25 & 827 \\ 25 & 59 \end{vmatrix}$. Subtracting the first row from the second row gives $\begin{vmatrix} 25 & 827 \\ 0 & -768 \end{vmatrix} = (25)(-768) = -19,200$.

(26.) Let $A \in M_{n\times n}(F)$. Under what conditions is $det(-A) = det(A)$?

Let's start out with a more general question. Suppose c is any scalar. Then to convert A to cA by elementary row operations, you multiply each row of A by c. Each one of those operations multiplies the determinant by c, so since A has n rows, we see $det(cA) = c^n det(A)$. Therefore, $det(cA) = det(A)$ when $c^n det(A) = det(A)$, which happens when $det(A) = 0$ or $c^n = 1$.

We can write $-A = (-1)A$, so $det(-A) = det(A)$ when $det(A) = 0$ or when $(-1)^n = 1$. Now $(-1)^n = 1$ when n is even.

However, $(-1)^n = 1$ also holds when $-1 = 1$; that is, when $1 + 1 = 0$. This can happen: remember back to our discussion of fields, when we looked at the field $\mathbb{Z}_2$ whose only elements are 0 and 1, and in which $1 + 1 = 0$. A field in which $1 + 1 = 0$ is said to have characteristic 2.

Putting it all together, $det(-A) = det(A)$ if and only if at least one of these three conditions holds:

1. n is even;
2. F has characteristic 2;
3. $det(A) = 0$.