Math 24
Spring 2012
Sample Homework Solutions
Week 5

Section 3.1

(1.) The answers are in the back of the book.

(2.)

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 3 \\ 1 & -2 & 1 \\ 1 & -3 & 1 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & -2 \\ 1 & -3 & 1 \end{pmatrix}$$

To convert A to B, add -2 times column 1 to column 2.
To convert B to C, add -1 times row 1 to row 3.
To convert C to I_3, add -1 times row 1 to row 3, add -3 times column 1 to column 3, multiply row 2 by $-\dfrac{1}{2}$, add 3 times row 2 to row 3 add -1 times column 2 to column 3.

(3.) Use the proof of Theorem 3.2 to obtain the inverse of each of these matrices. The proof of Theorem 3.2 shows that if I is converted to A by some row or column operation, then I is converted to A^{-1} by the inverse row or column operation.

(a.) $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. We can convert I to A by switching rows 1 and 3, and so we can convert I to A^{-1} by switching rows 1 and 3. $A^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

(b.) $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. We can convert I to A by multiplying column 2 by 3, and so we can convert I to A^{-1} by multiplying column 2 by $\frac{1}{3}$. $A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

(c.) $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$. We can convert I to A by adding -2 times row 1 to row 3, and so we can convert I to A^{-1} by adding 2 times row 1 to row 3. $A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$.

Section 3.2

(1.) The answers are in the back of the book.

(5f.) $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

To find the rank and inverse (if there is one) of A, we could try row-reducing the augmented matrix $(A|I)$. However, we can see that A has two identical columns, and the remaining column is not a multiple of these, so the number of linearly independent columns of A is 2, $rank(A) = 2$, and A is not invertible.

(6e.) Determine whether T is invertible, and if it is, compute T^{-1}.
$T : P_2(\mathbb{R}) \to \mathbb{R}^3$ is defined by $T(f) = (f(-1), f(0), f(1)))$.

Since $T(1) = (1,1,1)$, $T(x) = (-1,0,1)$, and $T(x^2) = (1,0,1)$, we can write down the matrix of T in the standard bases α for $P_2(\mathbb{R})$ and β for $\mathbb{R}^3$: $[T]_\alpha^\beta = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$. It is not too hard to see that the columns of the matrix are linearly independent, and so both the matrix and the linear transformation T are invertible. We can use the fact that $\left([T]_\alpha^\beta\right)^{-1} = [T^{-1}]_\beta^\alpha$ to find T^{-1}. First we invert our matrix to find $[T^{-1}]_\beta^\alpha$:

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \Longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \Longrightarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 & 1 \end{array}\right) \Longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 & 1 \end{array}\right) \Longrightarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 & -2 & 1 \end{array}\right) \Longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & -1 & \frac{1}{2} \end{array}\right) \Longrightarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & -1 & \frac{1}{2} \end{array}\right)$$

This tells us that $[T^{-1}]_\beta^\alpha = \begin{pmatrix} 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{pmatrix}$,

$$[T^{-1}(a,b,c)]_\alpha = \begin{pmatrix} 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} b \\ -\frac{1}{2}a + \frac{1}{2}c \\ \frac{1}{2}a - b + \frac{1}{2}c \end{pmatrix},$$

$$T^{-1}(a,b,c) = b + \left(-\tfrac{1}{2}a + \tfrac{1}{2}c\right)x + \left(\tfrac{1}{2}a - b + \tfrac{1}{2}c\right)x^2.$$

(7.) Express the invertible matrix $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ as the product of elementary matrices.

We can convert A to I by the following sequence of elementary row operations.

1. Add -1 times row 1 to row 2.
2. Add -1 times row 1 to row 3.
3. Add 1 times row 2 to row 1.
4. Add $-\frac{1}{2}$ times row 2 to row 3.
5. Multiply row 2 times $-\frac{1}{2}$.
6. Add -1 times row 3 to row 1.

Letting E_i stand for the elementary matrix corresponding to the i^{th} operation, and recalling that performing an elementary row operation is the same as multiplying on the left by the corresponding matrix, we see

$$E_6 E_5 E_4 E_3 E_2 E_1 A = I,$$

and therefore

$$A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} E_5^{-1} E_6^{-1}.$$

Since E_i^{-1} is the elementary matrix corresponding to the inverse of the i^{th} operation, we have

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$