Math 24
Spring 2012
Sample Homework Solutions
Week 2

Section 1.4

(5) In each part, determine whether the given vector is in the span of S.
(d) $(2, -1, 1, -3)$, $\quad S = \{(1, 0, 1, -1),\ (0, 1, 1, 1)\}$.
(f) $2x^3 - x^2 + x + 3$, $\quad S = \{x^3 + x^2 + x + 1,\ x^2 + x + 1,\ x + 1\}$.
(h) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\quad S = \left\{ \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\}$.

In each case we need to see whether the given vector can be written as a linear combination of the vectors in S. We can do this by expressing the given vector as linear combination of the vectors in S, with unknown coefficients, and solving for the coefficients. For example, in (d), we can write

$$(2, -1, 1, -3) = a(1, 0, 1, -1) + b(0, 1, 1, 1),$$

break this up coordinatewise to get a system of linear equations in a and b

$$a = 2 \quad b = -1 \quad a + b = 1 \quad -a + b = -3$$

and solve this system to get a solution, $a = 2, b = -1$.

(d) Yes, $(2, -1, 1, -3) = 2(1, 0, 1, -1) - (0, 1, 1, 1)$.

(f) No. We can see this because in each of the polynomials in S, the coefficient of the x term is the same as the constant term, so that will be true of every linear combination of those polynomials. Since it is not true of the given polynomial, that polynomial isn't in the span of S.

(h) No.

(10) Show that if

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

then the span of $\{M_1, M_2, M_3\}$ is the set of all symmetric 2×2 matrices.

Since M_1, M_2, and M_3 are all symmetric, every matrix in their span will be symmetric, so we need to show that every symmetric matrix is in their span. Every 2×2 symmetric matrix has the form

$$M = \begin{pmatrix} a & b \\ b & c \end{pmatrix},$$

and since we can write $M = aM_1 + cM_2 + bM_3$, any such M is in the span of $\{M_1, M_2, M_3\}$.

(12) Show that a subset W of a vector space V is a subspace of V if and only if $span(W) = W$.

We use the fact that $span(W)$ is the smallest subspace containing W.

If W is itself a subspace, then clearly W is the smallest subspace containing W, so $W = span(W)$.

On the other hand, if W is not a subspace, since $span(W)$ is a subspace we have $W \neq span(W)$.

Section 1.5

(2) Determine whether the following sets are linearly dependent or linearly independent.

(b) $\left\{ \begin{pmatrix} 1 & -2 \\ -1 & 4 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 2 & -4 \end{pmatrix} \right\}$ in $M_{2\times 2}(\mathbb{R})$.

(d) $\{x^3 - x,\ 2x^2 + 4,\ -2x^3 + 3x^2 + 2x + 6\}$ in $P_3(\mathbb{R})$.

(f) $\{(1, -1, 2),\ (2, 0, 2),\ (-1, 2, -1)\}$ in $\mathbb{R}^3$.

In each case we need to see whether there is a nontrivial linear combination of the given vectors equal to zero. As in problem (5) of section 1.4, in general this leads to solving a system of linear equations, although sometimes we can see the answer without doing that.

(b) We can see that neither of these matrices is a scalar multiple of the other, so this set is linearly independent.

(d) Because $4(x^3 - x) - 3(2x^2 + 4) + 2(-2x^3 + 3x^2 + 2x + 6) = 0$, this set is linearly dependent.

(e) This set is linearly independent. We can see that because the three vectors are not coplanar. (The second and third are not collinear but both lie in the plane $x = z$, and the first does not lie in that plane.)

(7) Recall from Example 3 in Section 1.3 that the set of diagonal matrices in $M_{2\times 2}(F)$ is a subspace. Find a linearly independent set that generates this subspace.

A natural choice is $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$.

(17) Let M be a square upper triangular matrix (as defined in Exercise 12 of Section 1.3) with nonzero diagonal entries. Prove that the columns of M are linearly independent.

$$M = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & a_{nn} \end{pmatrix}.$$

First, the columns of M are in fact vectors. We can write the vectors in $\mathbb{R}^n$ as either row vectors or column vectors, so the columns of M are vectors in $\mathbb{R}^n$.

To show the columns of M are linearly independent, consider a nontrivial linear combination (sum of scalar multiples) of the columns, and show it does not equal zero. Since the linear combination is nontrivial, some columns are multiplied by nonzero coefficients. Say k is the greatest number such that the coefficient of the k^{th} column is $b_k \neq 0$. Then, since every earlier column has a 0 as its k^{th} coordinate and every later column is multiplied by 0, the k^{th} coordinate of the linear combination is $b_k a_{kk}$, which is not zero.

Section 1.6

(4) Do the polynomials $x^3 - 2x^2 + 1$, $4x^2 - x + 3$, and $3x - 2$ generate $P_3(\mathbb{R})$? Justify your answer.

No. The dimension of $P_3(\mathbb{R})$ is four, so three polynomials cannot generate the entire space.

(7) The vectors $u_1 = (2, -3, 1)$, $u_2 = (1, 4, -2)$, $u_3 = (-8, 12, -4)$, $u_4 = (1, 37, -17)$, $u_5 = (-3, -5, 8)$ generate $\mathbb{R}^3$. Find a subset of $\{u_1, u_2, u_3, u_4, u_5\}$ that is a basis for $\mathbb{R}^3$.

If we follow the method in the text, we will start with u_1, since it is not zero. Then we will include u_2, since it is not a multiple of u_1. We need one more vector, and we will take the first one that is not a linear combination of u_1 and u_2. Now u_3 is a linear combination of u_1 and u_2 (in fact, it is a multiple of u_1), and so is u_4 (it's equal to $7u_2 - 3u_1$, which I found by solving the system of linear equations we get from setting $au_1 + bu_2 = u_4$). Since we are given that these vectors span the three-dimensional space $\mathbb{R}^3$, we know u_5 must not be a linear combination of u_1 and u_2, so a basis is $\{u_1, u_2, u_5\}$.

(11) Let u and v be distinct vectors of a vector space V. Show that if $\{u, v\}$ is a basis for V and a and b are nonzero scalars, then both $\{u+v, au\}$ and $\{au, bu\}$ are also bases for V.

Since $dim(V) = 2$ (which we know since $\{u, v\}$ is a basis), to show a set X containing two vectors is a basis, it is enough either to show that X is linearly independent or to show that X spans V.

We will show that $\{u+v, au\}$ spans V. Since $\{u, v\}$ spans V, it is enough to show that both u and v are in the span of $\{u+v, au\}$. This is true since $u = a^{-1}(au)$ and $v = (u+v) - u = (u+v) - a^{-1}(au)$.

Just for variety, we will show that $\{au, bv\}$ is linearly independent. To do this, suppose some linear combination equals zero, $c(au) + d(bv) = 0$. We must show that combination is trivial, that is, that $c = 0$ and $d = 0$.

We can rewrite our equation as $(ca)u + (db)v = 0$. Because u and v are linearly independent, we must have $ca = 0$ and $db = 0$. Because a and b are nonzero, we can multiply by a^{-1} and b^{-1} respectively, to get $c = 0$ and $d = 0$.

(13) The set of solutions to the system of linear equations

$$x_1 - 2x_2 + x_3 = 0$$

$$2x_1 - 3x_2 + x_3 = 0$$

is a subspace of $\mathbb{R}^3$. Find a basis for this subspace.

Applying Gaussian elimination to this system, we get the equivalent system

$$x_1 - x_3 = 0$$

$$x_2 - x_3 = 0.$$

The first equation lets us solve for x_1, yielding $x_1 = x_3$, and the second equation lets us solve for x_2, yielding $x_2 = x_3$. We have no equation corresponding to x_3, so we introduce a parameter for x_3, setting $x_3 = s$ and $(x_1, x_3, x_3) = (s, s, s) = s(1, 1, 1)$.

This means every solution is in this form, so the subspace of solutions consists of all vectors of the form $s(1, 1, 1)$, or all multiples of the vector $(1, 1, 1)$. A basis for this subspace is $\{(1, 1, 1)\}$.