

Math 24
Winter 2010
Special Assignment due Monday, February 1

Let V be any vector space and W be a subspace of V . For any vector x in V , we define the *coset* of W containing x to be

$$x + W = \{x + w \mid w \in W\}.$$

We denote the collection of cosets of W in V by V/W :

$$V/W = \{x + W \mid x \in V\}.$$

For your last assignment, you proved that addition of cosets defined by

$$(x + W) + (y + W) = (x + y) + W$$

is well-defined.

Assignment: Prove that V/W , with addition defined as above, satisfies the first four vector space axioms.

Note: (This note is just cultural enrichment. You can ignore it, or read it later.) We can make a similar definition for other sorts of structures and substructures. For example, the integers \mathbb{Z} with addition and multiplication form a “commutative ring with unity.” This is a structure that satisfies all the axioms for a field except possibly the existence of multiplicative inverses. The set of multiples of n

$$n\mathbb{Z} = \{nx \mid x \in \mathbb{Z}\}$$

is a kind of substructure of \mathbb{Z} called an “ideal.” This means it is closed under addition, and also under multiplication by any element of \mathbb{Z} . Now if we define cosets of $n\mathbb{Z}$ the same way we did above,

$$x + n\mathbb{Z} = \{x + m \mid m \in n\mathbb{Z}\},$$

we can define addition and multiplication of cosets

$$(x + n\mathbb{Z}) + (y + n\mathbb{Z}) = (x + y) + n\mathbb{Z} \quad \text{and} \quad (x + n\mathbb{Z})(y + n\mathbb{Z}) = (xy) + n\mathbb{Z}.$$

We get the structure $\mathbb{Z}/n\mathbb{Z}$, whose elements are cosets $0 + n\mathbb{Z}$, $1 + n\mathbb{Z}$, \dots , $(n - 1) + n\mathbb{Z}$.

$\mathbb{Z}/2\mathbb{Z}$ is the same as \mathbb{Z}_2 (defined in Appendix C of the textbook), except that instead of calling the elements $0 + 2\mathbb{Z}$ and $1 + 2\mathbb{Z}$, the textbook just calls them 0 and 1. Another name for $\mathbb{Z}/n\mathbb{Z}$ is “the integers modulo n .” If you’re up for a challenge, you might notice that while $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$ are fields, $\mathbb{Z}/4\mathbb{Z}$ is not. For which n is $\mathbb{Z}/n\mathbb{Z}$ a field?