Math 24

Spring 2012

Special Assignment due Monday, April 16

Let V be any vector space and W be a subspace of V. For any vector x in V, we define the *coset* of W containing x to be $x + W = \{x + w \mid w \in W\}$.

For your last assignment, you proved that for any vector space V and any subspace W of V,

1. For any x in V, $x \in (x + W)$;
2. For any x and y in V, $(x - y) \in W \implies (x + W) = (y + W)$;
3. For any x and y in V, $(x - y) \notin W \implies (x + W) \cap (y + W) = \emptyset$.

That is, the cosets of W form a *partition* of V; different (unequal) cosets are disjoint (do not overlap), and together, the cosets cover all of V. In the example given in the assignment, $V = \mathbb{R}^2$ and W is the x-axis, and the cosets of W are all the lines parallel to the x-axis: $(a, b) + W$ is the line $y = b$.

Now we will define a method of adding cosets to each other.

Definition: If $X = x + W$ and $Y = y + W$ are two cosets of W, we define their sum to be

$$X + Y = (x + y) + W.$$

We could also say simply,

$$(x + W) + (y + W) = (x + y) + W.$$

In the given example, $(1, 2) + W$ is the line $y = 2$, and $(1, -1) + W$ is the line $y = -1$. Then by our definition,

$$\big((1, 2) + W\big) + \big((1, -1) + W\big) = \big((1, 2) + (1, -1)\big) + W = (2, 1) + W,$$

which is the line $y = 1$.

For this definition to make sense, we have to know that if, for example, we think of the line $y = 2$ as the coset $(3, 2) + W$, and the line $y = -1$ as the coset $(-5, -1) + W$, and use this definition to add those two lines, we get the same answer. It's easy to check in this specific case; our definition would say

$$\big((3, 2) + W\big) + \big((-5, -1) + W\big) = \big((3, 2) + (-5, -1)\big) + W = (-2, 1) + W,$$

which is indeed the line $y = 1$.

We need to know that this always happens. If it does we say that addition of cosets is *well-defined*.

Assignment: Show that addition of cosets is well-defined.

That is, show that whenever we have

$$X = x + W = x' + W \text{ and } Y = y + W = y' + W$$

then we have

$$(x + y) + W = (x' + y') + W.$$

Note: You may be tempted to say the following.

> Suppose that $x + W = x' + W$ and $y + W = y' + W$;
> then we must show that $(x + W) + (y + W) = (x' + W) + (y' + W)$.

Do not do that. You should not use the expression $(x' + W) + (y' + W)$ until you know that addition of cosets makes sense. If you can come up with two different answers for $X + Y$ depending on how you write X and Y, then addition of cosets does not make sense. You are still trying to prove that can't happen. Say instead,

> Suppose that $x + W = x' + W$ and $y + W = y' + W$;
> then we must show that $(x + y) + W = (x' + y') + W$.

Note: Here is an example of a proposed operation that is not well-defined. For subspaces W of $\mathbb{R}^n$ (where we have the notion of dot product), suppose we try to define the dot product of two cosets by

$$(x + W) \cdot (y + W) = (x \cdot y).$$

Then, in the same example as before, if we take $X = (1, 2) + W = (3, 2) + W$ and we take $Y = (1, -1) + W = (-5, -1) + W$, then we have

$$(1, 2) \cdot (1, -1) = -1 \text{ and } (3, 2) \cdot (-5, -1) = -17,$$

so $X \cdot Y$ would have to equal both -1 and -17. That means this notion of the dot product of two cosets is NOT well-defined, and we can't use this definition.

The moral: If you are defining some function of X, if your definition depends on how you name or express X, and if there can be more than one way to name or express the same X, then you must verify that your function is well-defined. If it is not, then you have not actually defined a function.

The same thing applies if there is some other reason that your definition might assign more than one value to some X. For example, if we tried to define a function $f : \mathbb{C} \to \mathbb{C}$ by setting $f(x)$ to be the number y such that $y^2 = x$, we would not have a well-defined function. This is because (unless x is 0) there are not one but two complex numbers y such that $y^2 = x$. Since the definition does not identify one and only one value for $f(x)$, the function is not well-defined.

One more note (for those who like subtleties): The vector space axiom VS4 (the existence of additive inverses) says, "For each element x in V there exists an element y in V such that $x + y = 0$." Not until after the proof of Corollary 2 to Theorem 1.1 (namely, that this vector y is unique) does the textbook then define $-x$ to be this vector y. Before the proof of Corollary 2, we would not have known that $-x$ (the additive inverse of x) is well-defined.

If you now look back at axiom VS3, things get really subtle. This axiom reads, "There exists an element in V denoted by 0 such that $x + 0 = x$ for each x in V." That squeaks by as legitimate; it says there is an additive identity denoted by 0, without claiming there is no other additive identity. (The fact that there is no other additive identity must be proved, and it is proved as Corollary 1 to Theorem 1.1.) If the axiom instead read, "There exists an element y in V such that $x + y = x$ for each x in V. We denote this element by 0," that would be wrong. We would not (yet) know that "0" is well-defined.

We could instead phrase axiom VS3 as, "There exists a unique element y in V such that $x+y = x$ for each x in V. We denote this element by 0." That would be legitimate. However, it is better to have the axiom as it is, and prove that the additive identity is unique. The reason this is better is that if the axiom said, "There exists a unique element y in V such that $x + y = x$ for each x in V," then whenever we wanted to prove something is a vector space, we would need to prove not only that it has an additive identity, but also that it has only one additive identity.