Math 24
Spring 2012

Sample Quiz Solutions

Monday, April 16

1. $\boxed{TRUE}$ or FALSE: It is possible for a linear transformation from $\mathbb{Q}^3$ to $M_{2\times 2}(\mathbb{Q})$ to be one-to-one but not onto.

   We know this because the domain has dimension 3 and the codomain has dimension 4.

2. A linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^3$ is defined by

$$T(x, y, z, w) = \big((x + y + z),\ (y + z + w),\ (x - w)\big).$$

   Give a basis for the null space of T.

   The null space is the set of vectors (x, y, z, x) satisfying the system of linear equations

$$x + y + z = 0$$

$$y + z + w = 0$$

$$x - w = 0,$$

   which can be converted by Gaussian elimination to

$$x - w = 0$$

$$y + z + w = 0$$

$$0 = 0.$$

   Using the first two equations to solve for x and y, and introducing parameters s for z and t for w, we get the general solution

$$(x, y, z, w) = (t, -s - t, s, t).$$

We can find a basis by setting $(s,t) = (1,0)$ and $(s,t) = (0,1)$. A basis is

$$\boxed{\{(0,-1,1,0,),\ (1,-1,0,1)\}}.$$

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$$T(x,y,z,w) = (x+y+z,\ y+z+w,\ x-w).$$

This is the same linear transformation as in the preceding problem.

3. $r(T) = \boxed{2}$.

   From the dimension theorem, since the dimension of the domain is 4 and the dimension of the null space is 2, the dimension of the range must be $4-2$, or 2.

   (The original problem mistakenly read $R(T)$ instead of $r(T)$. The range of T is the span of the images of the basis vectors of the domain; you could have expressed $R(T)$ as the span of $(1,0,1)$ $(1,1,0)$, $(1,1,0)$, and $(0,1,-1)$.)

4. If β is the standard basis for $\mathbb{R}^4$ and γ is the standard basis for $\mathbb{R}^3$,

   $$[T]_\beta^\gamma = \boxed{\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix}}.$$

   The columns of this matrix are the coordinates of the images of the basis vectors of $\mathbb{R}^4$.