Math 24
Spring 2012
Some Proof Principles

Generally, proving something requires some creativity; there is no recipe for producing a proof. However, there are some standard techniques that can be used, depending on the form of the statement you are trying to prove. (Note that "can" does not mean "must.") Here are a few of them.

1. To prove a statement of the form "If A, then B," assume A and prove B. Or, prove the *contrapositive*: "If not B, then not A," by assuming not B and proving not A.

2. To prove a statement of the form "not A," use *proof by contradiction*: Assume A, and deduce a contradiction, something obviously false or contradictory.

3. To prove a statement of the form "For all vectors x, A(x)," let x be a name for an arbitrary vector, and prove A(x).

4. To prove a statement of the form "There is a vector x such that A(x)," find a specific example $\vec{v}$ and prove that A($\vec{v}$). (For example, prove that A($\vec{0}$).)

5. To prove a statement of the form "A and B," prove both A and B.

6. To prove a statement of the form "A or B," prove "If not A, then B," or prove "If not B, then A," or assume "Not A and not B" and deduce a contradiction. Or, consider all possible cases, and prove that in some cases A holds, and in other cases B holds.

7. In general, prove something by considering all possible cases separately. You must be sure the cases you list cover all possibilities. For an example of a proof like this, see the next page.

8. To prove something is unique, assume there are two such things, and prove they are actually equal.

9. To prove a statement of the form "There is a unique x such that A(x)," prove both "There is an x such that A(x)" and "the x such that A(x) is unique." This is called proving existence and uniqueness.

Proposition: Suppose that $X \subseteq \mathbb{R}^2$ is a subspace of $\mathbb{R}^2$. (That is, X, with the same addition and scalar multiplication as in $\mathbb{R}^2$, is itself a vector space over $\mathbb{R}$.) Then X must be one of

1. The zero vector space, $\{\vec{0}\}$.
2. A line through the origin.
3. All of $\mathbb{R}^2$.

Proof: There are three possible cases for X:

1. X contains no nonzero vectors.
2. X contains at least one nonzero vector, and all nonzero vectors in X are parallel.
3. X contains at least one pair of nonzero vectors that are not parallel.

We consider each case separately.

1. X must contain at least one vector, by vector space axiom (VS 3). Therefore, since X does not contain any nonzero vectors, X must contain the zero vector, and we have $X = \{\vec{0}\}$. That is, X is the zero vector space.
2. Let $\vec{v}$ be some nonzero element of X. If $\vec{w}$ is any other element of X, either $\vec{w} = \vec{0}$ or $\vec{w}$ is parallel to $\vec{v}$; in either case, $\vec{w}$ is a scalar multiple of $\vec{v}$, that is, $\vec{w} = t\vec{v}$ for some scalar t.

   Now, because X is a vector space, X is closed under multiplication by scalars, so *every* scalar multiple of $\vec{v}$ must be in X. Therefore X must consist exactly of all the scalar multiples of $\vec{v}$,
   $$X = \{t\vec{v} \mid t \in \mathbb{R}\}.$$
   That is, X is the line through the origin in the direction of $\vec{v}$.
3. Let $\vec{v}$ and $\vec{w}$ be nonzero, nonparallel elements of X. Because X is closed under both addition and multiplication by scalars, every vector of the form $s\vec{v} + t\vec{w}$ must be in X. To show $X = \mathbb{R}^2$, we must show every vector $(c_1, c_2) \in \mathbb{R}^2$ can be written in the form $s\vec{v} + t\vec{w}$.

   Method 1: Argue geometrically. Since $\vec{v}$ and $\vec{w}$ are not parallel, you can get from $(0, 0)$ to any point in the plane by proceeding some distance in the direction of $\vec{v}$ and then some distance in the direction of $\vec{w}$. That is, you can express any element of $\mathbb{R}^2$ as the sum of a scalar multiple of $\vec{v}$ and a scalar multiple of $\vec{w}$.

   Method 2: Argue algebraically.

Suppose $\vec{v} = (a_1, a_2)$ and $\vec{w} = (b_1, b_2)$. We must show that for any choice of (c_1, c_2) we can find real numbers s and t such that

$$s(a_1, a_2) + t(b_1, b_2) = (c_1, c_2).$$

That is, we must show we can always solve the system of linear equations

$$a_1 s + b_1 t = c_1$$

$$a_2 s + b_2 t = c_2$$

for s and t.

(Note: I was able to come up with the following argument because I already know linear algebra. It uses Cramer's Rule, page 224 of the textbook. You might come up with a similar argument by trying to solve the system of linear equations, and seeing what you need to assume in order to solve it.)

We claim that if $a_1 b_2 = a_2 b_1$, then $\vec{v}$ and $\vec{w}$ are parallel. Check this by cases:

(a) If $a_1 = 0$, then $a_1 b_2 = 0$, so by assumption $a_2 b_1 = 0$. Since $(a_1, a_2) = \vec{v} \neq (0, 0)$, we must have $a_2 \neq 0$, and so $b_1 = 0$. In this case, $\vec{v} = (0, a_2)$ and $\vec{w} = (0, b_2)$ are parallel.

(b) If $a_2 = 0$, a similar argument shows $\vec{v}$ and $\vec{w}$ are parallel.

(c) If $a_1 \neq 0$ and $a_2 \neq 0$, we can divide $a_1 b_2 = a_2 b_1$ by $a_1 a_2$ to get

$$\frac{b_2}{a_2} = \frac{b_1}{a_1} = d,$$

from which we have

$$d(a_1, a_2) = (da_1, da_2) = \left(\frac{b_1}{a_1} a_1, \frac{b_2}{a_2} a_2\right) = (b_1, b_2),$$

showing $\vec{v}$ and $\vec{w}$ are parallel.

Since $\vec{v}$ and $\vec{w}$ are not parallel, $a_1 b_2 \neq a_2 b_1$, and so $a_1 b_2 - a_2 b_1 \neq 0$, In that case, we can check that

$$s = \frac{c_1 b_2 - c_2 b_1}{a_1 b_2 - a_2 b_1} \qquad t = \frac{a_1 c_2 - a_2 c_1}{a_2 b_2 - a_2 b_1}$$

is a solution of the system of linear equations.