Math 24
Spring 2012
Friday, April 13

1. Assume $V$ and $W$ are finite-dimensional vector spaces with ordered bases $\beta$ and $\gamma$ respectively, and $T$ and $U$ are linear transformations from $V$ to $W$.

   TRUE or FALSE?

   (a) For any scalar $a$, $aT + U$ is a linear transformation from $V$ to $W$. (T)
   (b) $[T]_\beta^\gamma = [U]_\beta^\gamma$ implies that $T = U$. (T)
   (c) If $m = dim(V)$ and $n = dim(W)$ then $[T]_\beta^\gamma$ is an $m \times n$ matrix. (F) (It's $n \times m$.)
   (d) $[T + U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$. (T)
   (e) $\mathcal{L}(V, W) = \mathcal{L}(W, V)$. (F) (They do have the same dimension, $mn$.)
   (f) If $m = dim(V)$ then the function $f : V \to F^m$ defined by $f(v) = [v]_\beta$ is a linear transformation from $V$ to $F^m$. (T)
   (g) Every element of $M_{3\times3}(\mathbb{R})$ is the matrix of some linear transformation from $\mathbb{R}^3$ (with the standard ordered basis) to $P_2(\mathbb{R})$ (with the standard ordered basis). (T)
   (h) If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation such that $[T]_\beta^\beta = \begin{pmatrix} 2 & 4 \\ 1 & 0 \end{pmatrix}$, where $\beta$ is the standard ordered basis for $\mathbb{R}^2$, then $T(1, -1) = (1, 4)$. (F)

Here is the computation from that last item. The standard ordered basis is $\beta = \{e_1, e_2\} = \{(1, 0), (0, 1)\}$.

The columns of the matrix, therefore, are the coordinates of the images the vectors in $\beta$, that is, the coordinates of $T(e_1)$ and $T(e_2)$.

Because the ordered basis of the codomain is $\beta$, when we say "coordinates" in the previous sentence, we mean "$\beta$ coordinates." That is, we see from $[T]_\beta^\beta$ that

$$[T(e_1)]_\beta = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{and} \quad [T(e_2)]_\beta = \begin{pmatrix} 4 \\ 0 \end{pmatrix}.$$

Because the designated codomain basis vectors are $e_1$ and $e_2$, this tells us that

$$T(e_1) = 2e_1 + 1e_2 = (2, 1) \quad \text{and} \quad T(e_2) = 4e_1 + 0e_2 = (4, 0).$$

Therefore, because $T$ is linear, we can write

$$T(1, -1) = T(e_1 - e_2) = T(e_1) - T(e_2) = (2, 1) - (4, 0) = (-2, 1).$$

2. Let $T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$ be defined as follows:

If $p(x)$ is a polynomial in $P_2(x)$, then $T(p(x))$ is the antiderivative $q(x)$ of $p(x)$ such that $q(0) = 0$. Another way to say this is

$$T(p(x)) = \int_0^x p(t)\,dt.$$

If $\beta = \{1, x, x^2\}$ and $\gamma = \{1, x, x^2, x^3\}$ are the standard bases for $P_2(\mathbb{R})$ and $P_3(\mathbb{R})$, find the matrix $[T]_\beta^\gamma$.

The columns of $[T]_\beta^\gamma$ are the $\gamma$-coordinates of the images of the basis vectors in $\beta$.

The basis vectors in $\beta$ are $\{1, x, x^2\}$.

Their images are $T(1) = x$, $T(x) = \frac{1}{2}x^2$, $T(x_2) = \frac{1}{3}x^3$.

To find the columns of the matrix, we write these vectors out as linear combinations of the vectors in $\gamma$, which gives us their $\gamma$-coordinates:

$T(1) = x = (0)1 + (1)x + (0)x^2 + (0)x^3$,
$T(x) = \frac{1}{2}x^2 = (0)1 + (0)x + \frac{1}{2}x^2 + (0)x^3$,
$T(x^2) = \frac{1}{3}x^3 = (0)1 + (0)x + (0)x^2 + \frac{1}{3}x^3$.

$[T(1)]_\gamma = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $[T(x)]_\gamma = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix}$, $[T(x^2)]_\gamma = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{3} \end{pmatrix}$.

These coordinate vectors are the columns of the matrix.

$[T]_\beta^\gamma = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$.

3. Let $\beta = \{(1,0),(0,1)\}$ and $\gamma = \{(1,1),(1,-1)\}$ be two ordered bases for $\mathbb{R}^2$.

(a) Find:

$$[(1,1)]_\beta = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$[(1,-1)]_\beta = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

(b) Find:

$$[(1,0)]_\gamma = \begin{pmatrix} \frac{1}{2} \\ \\ \frac{1}{2} \end{pmatrix}$$

$$[(0,1)]_\gamma = \begin{pmatrix} \frac{1}{2} \\ \\ -\frac{1}{2} \end{pmatrix}$$

(c) Recall that $I$ denotes the identity function, $I(\vec{v}) = \vec{v}$. Find the matrices:

$$[I]_\beta^\beta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$[I]_\gamma^\gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$[I]_\beta^\gamma = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

$$[I]_\gamma^\beta = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

4. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ and $U : \mathbb{R}^2 \to \mathbb{R}^2$ be linear transformations. Let $\beta = \{(1,0),\ (0,1)\}$ be the standard ordered basis for $\mathbb{R}^2$.

   (a) If $T(1,0) = (a,c)$ and $T(0,1) = (b,d)$, then find:

   $T(x,y) = (ax + by,\ cx + dy)$

   $[T]_\beta^\beta = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

   (b) If $U(x,y) = (\overline{a}x + \overline{b}y,\ \overline{c}x + \overline{d}y)$, then find :

   $U(1,0) = (\overline{a},\ \overline{c})$

   $U(0,1) = (\overline{b},\ \overline{d})$

   $[U]_\beta^\beta = \begin{pmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{pmatrix}$

   (c) The composition of $U$ and $T$ is denoted $T \circ U$, or simply $TU$, and is defined by $TU(x,y) = T(U(x,y))$. For $T$ as in part (a) and $U$ as in part (b), find:

   $TU(1,0) = T(\overline{a},\ \overline{c}) = (a\overline{a} + b\overline{c},\ c\overline{a} + d\overline{c})$

   $TU(0,1) = T(\overline{b},\ \overline{d}) = (a\overline{b} + b\overline{d},\ c\overline{b} + d\overline{d})$

   $[TU]_\beta^\beta == \begin{pmatrix} a\overline{a} + b\overline{c} & a\overline{b} + b\overline{d} \\ c\overline{a} + d\overline{c} & c\overline{b} + d\overline{d} \end{pmatrix}$

In the next section of the text, you will see matrix multiplication defined. *This is where the definition comes from.* Matrix multiplication is defined so that if $A$ is the matrix of $T$ and $B$ is the matrix of $U$, then $AB$ is the matrix of $TU$. You have just come up with the formula for the product of two $2 \times 2$ matrices.

5. This an extra problem, not really part of our Math 24 study.

If we think of function composition as a kind of multiplication, then if $V$ is a vector space over a field $F$, the collection $\mathcal{L}(V)$ of linear transformations from $V$ to itself has an addition operation and a multiplication operation. (We know that $\mathcal{L}(V)$ is closed under these operations; the sum of linear functions is linear and the composition of linear functions is linear.)

With these two operations, is $\mathcal{L}(V)$ a field? If not, which of the field axioms hold, and which do not?

I'll leave this one as a challenge.

This algebraic structure turns out to obey many of the field axioms, but not all of them. It is not a field but it is a *ring*. The ring axioms include many but not all of the field axioms.

The field axioms:

(F1) (a) Adddition is commutative.

(b) Multiplication is commutative.

(F2) (a) Adddition is associative.

(b) Multiplication is associative.

(F3) (a) There is an additive identity element.

(b) There is a multiplicative identity element.

(F4) (a) Every element has an additive inverse.

(b) Every element except the additive identity has an multiplicative inverse.

(F5) Multiplication distributes over addition.