Math 24
Spring 2012
Wednesday, April 4
Sample Solutions

1. Find a set of linearly independent vectors whose span is the plane in $\mathbb{R}^3$ with equation
$$3x - 2y + z = 0.$$

 Sample Solution:

 Since this subspace is a plane, we need two non-collinear vectors to span it. One way to find them is by inspection. For example, we can rewrite the equation as $z = 2y - 3x$, then set one of x and y to 1 and the other to 0 to get solutions $(1, 0, -3)$ and $(0, 1, 2)$.

 A General Method:

 In general we might have m equations $a_1x_1 + \cdots + a_nx_n = 0$ in n variables, and ask for a linearly independent set spanning the collection of vectors $(x_1, x_2, \ldots, x_n)$ whose entries x_1, x_2, ..., x_n are solutions of the system. A general method that works begins by using Gaussian elimination to solve the system.

 > For our (very simple) system, we identify the first equation as the x-equation, and divide through by 3 to get an x-coefficient of 1:
 > $$\boxed{x} - \frac{2}{3}y + \frac{1}{3}z = 0.$$
 > If there were more equations, we would eliminate x in the other equations, then go on to identify a y-equation, if possible, and so on. Since there aren't, we are done with this part.

 We will have equations for some variables but (possibly) not for others. We introduce *parameters* for the variables without equations, and solve for the other variables by using their equations.

In our example, since there is no y-equation and no z-equation, y and z can be anything, so we introduce *parameters*, setting $y = s$ and $z = t$. Then we use the x-equation to solve for x, getting $x = \frac{2}{3}y - \frac{1}{3}z$, and so we have

$$(x, y, z) = \left(\frac{2}{3}s - \frac{1}{3}t,\ s,\ t\right) = s\left(\frac{2}{3},\ 1,\ 0\right) + t\left(-\frac{1}{3},\ 0,\ 1\right).$$

Now every way of setting one of our parameters to 1 and the rest to 0 gives a vector in the solution space. The method guarantees that these vectors span the space.

In our example, we first set $(s, t) = (1, 0)$ and then set $(s, t) = (0, 1)$ to get our two vectors,

$$\left\{\left(\frac{2}{3},\ 1,\ 0\right),\ \left(-\frac{1}{3},\ 0,\ 1\right)\right\}.$$

2. Give an example of three linearly dependent vectors in $\mathbb{R}^3$, none of which is a multiple of any other.

   Sample Solution:

   Because our three vectors must be linearly dependent, they must lie in a single plane containing the origin; because none can be a multiple of any other, no two can be parallel.

   An easy solution is to pick three vectors in the xy-plane, no two of which are parallel, such as

$$\{(1, 0, 0),\ (0, 1, 0),\ (1, 1, 0)\}.$$

   It will always work to take two vectors $\vec{u}$ and $\vec{v}$, neither of which is a multiple of the other, and take $\vec{u} + \vec{v}$ as your third vector.

3. Find a linearly independent subset of

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\}$$

with the same span.

Sample Solution:

We can see by inspection that the second matrix is the sum of the first and third, so we can eliminate it without changing the span of the set. This gives us

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\}.$$

This set is linearly independent. We can see that because if a linear combination of its elements equals the zero matrix, then the coefficient of the second matrix must be 0 (otherwise there would be a nonzero entry in the lower left), therefore the coefficient of the third matrix must be 0 (otherwise there would be a nonzero entry in the upper right), therefore the coefficient of the first matrix must be 0.

"By inspection" and "we can see" means we're lucky these matrices are simple enough so we can just look at them and tell whether they're linearly independent. Suppose they weren't?

Let's call the given matrices M_1, M_2, M_3, M_4. We can check for linear independence by finding all solutions to

$$a_1M_1 + a_2M_2 + a_3M_3 + a_4M_4 = 0.$$

If the only solution is $a_1 = a_2 = a_3 = a_4 = 0$, then our set is linearly independent. If not, say we see that

$$M_1 - M_2 + M_3 = 0,$$

this tells us we can write M_2 as a linear combination of M_1 and M_3, therefore we can eliminate it from the set without changing the span.

Now we apply the same technique to M_1, M_3, M_4 to either see that this set is linearly independent, or find another element we can eliminate.

Since the set gets smaller every time, eventually the process will come to an end, giving us a linearly independent set with the same span.

4. Show that if $\vec{u}$ and $\vec{v}$ are distinct elements of the vector space V, then $\{\vec{u}, \vec{v}\}$ is linearly dependent if and only if one of $\vec{u}$, $\vec{v}$ is a scalar multiple of the other.

   Sample Solution:

   If one of the vectors is a scalar multiple of the other, say $a\vec{v} = \vec{u}$, then we have $\vec{u} - a\vec{v} = \vec{0}$, a nontrivial linear combination equal to zero, so $\{\vec{u}, \vec{v}\}$ is linearly dependent.

   Conversely, suppose $\{\vec{u}, \vec{v}\}$ is linearly dependent. This means there is a nontrivial linear combination equal to zero, $a\vec{u} + b\vec{v} = \vec{0}$, where a and b are not both zero. Without loss of generality, $a \neq 0$. Then we can rewrite our equation as $a\vec{u} = -b\vec{v}$, or

$$\vec{u} = \left(\frac{-b}{a}\right)\vec{v}.$$

> Comment: The *converse* of "if A then B" is "if B then A." So "conversely" means that we have just finished proving one direction of an "if and only if" statement, and are about to prove the other.
>
> "Without loss of generality" is a very useful phrase. It means that it looks like we are considering only one case, but that's okay, because all cases work the same way. In this example, we don't have to consider the case $b \neq 0$, because the proof is exactly the same as in the case $a \neq 0$.
>
> You may notice I tried to slip the same thing by in the first part of the proof, by writing "say $a\vec{v} = \vec{u}$," implying that if we said $a\vec{u} = \vec{v}$ the proof would be the same.

5. Show that a set of vectors $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n\}$ is linearly dependent if and only if either $\vec{x}_1 = \vec{0}$, or there is some $k < n$ such that $\vec{x}_{k+1}$ is a linear combination of the vectors $\vec{x}_1, \ldots, \vec{x}_k$.

   Note: The point of this result is that, if you want to build a linearly independent set of vectors, as long as you start with a nonzero vector, keep adding vectors one at a time, and never add a vector that is a linear combination of the ones you already have, then you can't go wrong.

   Sample Solution:

   One direction is easy: If $\vec{x}_1 = \vec{0}$ then the set is linearly dependent (any set containing the zero vector is linearly dependent). If we can write some $\vec{x}_{k+1}$ as a linear combination of the vectors $\vec{x}_1, \ldots, \vec{x}_k$,

$$\vec{x}_{k+1} = a_1\vec{x}_1 + \cdots + a_k\vec{x}_k,$$

   then also the set is linearly dependent, because there is a nontrivial linear combination of vectors from the set that equals zero,

$$\vec{x}_{k+1} - a_1\vec{x}_1 - \cdots - a_k\vec{x}_k = \vec{0}.$$

   For the other direction, assume the set is linearly dependent, and show that either $\vec{x}_1 = \vec{0}$ or we can write some $\vec{x}_{k+1}$ as a linear combination of vectors $\vec{x}_1, \ldots, \vec{x}_k$.

   To do this, suppose the set is linearly dependent. Therefore there is some nontrivial linear combination that is equal to zero,

$$a_1\vec{x}_1 + \cdots + a_n\vec{x}_n = \vec{0},$$

   where not all the scalars a_i equal 0.

   If a_1 is the only nonzero scalar, we have $a_1\vec{x}_1 = \vec{0}$, and we can multiply both sides by $(a_1)^{-1}$ to get $\vec{x}_1 = \vec{0}$.

   If not, let a_{k+1} be the last nonzero scalar, so we have

$$a_1\vec{x}_1 + a_2\vec{x}_2 + \cdots + a_{k+1}\vec{x}_{k+1} = \vec{0},$$

where $a_{k+1} \neq 0$. we can rewrite this equation as

$$a_{k+1}\vec{x}_{k+1} = -a_1\vec{x}_1 - a_2\vec{x}_2 - \cdots - a_k\vec{x}_k,$$

and then multiply both sides by $(a_{k+1})^{-1}$ to write $\vec{x}_{k+1}$ as a linear combination of $\vec{x}_1, \ldots, \vec{x}_k$.

Another Possibility:

We can show that if $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n\}$ is linearly dependent, then either $\vec{x}_1 = \vec{0}$, or there is some $k < n$ such that $\vec{x}_{k+1}$ is a linear combination of the vectors $\vec{x}_1, \ldots, \vec{x}_k$, by *induction* on n.

> Comment: We say induction "on n" because we are proving that "for every n, ...," and our base case and inductive step deal with different values of n. If there is more than one relevant integer, it's important to specify which one you are doing induction on.

Base Case: ($n = 1$) We have already seen that if $\{\vec{x}_1\}$ is linearly dependent, then $\vec{x}_1 = \vec{0}$.

> Comment: When I outlined proof by induction in class, I said the base case was $n = 0$. Here I use $n = 1$, because the statement of the theorem only applies to sets with at least 1 element. Your base case is the smallest number you are interested in.
>
> The base case is often (but not always) particularly easy.

Inductive Step: Assume this is true for a linearly independent set of size n, and show it is true for a linearly independent set of size $n + 1$.

> Comment: "Assume this is true" sounds like assuming what you are trying to prove, but it really isn't. You have just proved the theorem holds for $n = 1$, so you know it's true for some numbers. Here, we are saying, "Assume that n is a number for which the theorem holds, and show that $n + 1$ is another number for which the theorem holds."
>
> In other words, we are showing the set of numbers for which the theorem holds is *closed* under adding 1 — if n is in, so

is $n+1$. Then we use the fact that a set that contains 1 and is closed under adding 1 must contain every positive integer, to conclude that the theorem holds for all positive integers.

Terminology: The hypothesis of the inductive step, in our case "the theorem is true for a linearly independent set of size n," is called the *inductive hypothesis*.

Suppose, then, that $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n, \vec{x}_{n+1}\}$ is linearly dependent.

Case 1: Suppose $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n\}$ is linearly dependent. Then by inductive hypothesis, either $\vec{x}_1 = \vec{0}$, or for some $k < n$, we can write $\vec{x}_{k+1}$ as a linear combination of $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k$.

Comment: "By inductive hypothesis" is common phrasing in proofs by induction. The reader is supposed to know this means "because we assumed the theorem holds for n."

Case 2: Suppose $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n\}$ is linearly independent. Because the entire set is linearly dependent, there is a nontrivial linear combination of its elements that equals $\vec{0}$,

$$a_1\vec{x}_1 + a_2\vec{x}_2 + \cdots + a_n\vec{x}_n + a_{n+1}\vec{x}_{n+1} = \vec{0},$$

where not all a_i are zero. It must be the case that $a_{n+1} \neq 0$, because otherwise we would have a nontrivial linear combination

$$a_1\vec{x}_1 + a_2\vec{x}_2 + \cdots + a_n\vec{x}_n = \vec{0},$$

which can't happen because $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$ are linearly independent. Therefore we can express $\vec{x}_{n+1}$ as a linear combination of $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$:

$$\vec{x}_{n+1} = \frac{-a_1}{a_{n+1}}\,\vec{x}_1 + \frac{-a_2}{a_{n+1}}\,\vec{x}_2 + \cdots + \frac{-a_n}{a_{n+1}}\,\vec{x}_n.$$

This completes the proof.

Another example of a proof by mathematical induction. We will prove this result later in the textbook.

Proposition: If

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = d_1,$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = d_2,$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = d_m,$$

is a system m linear equations in n variables over an infinite field F, and $m < n$, then the system has either no solutions or infinitely many.

Proof: By induction on m.

Base Case: If $m = 1$ then we have one equation

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = d_1,$$

and n is at least 2.

If $a_{1i} = 0$ for all i, and $d_1 = 0$, then any values of the x_i constitute a solution, so there are infinitely many solutions.

If $a_{1i} = 0$ for all i, and $d_1 \neq 0$, then there are no solutions.

If some $a_{1i} \neq 0$, without loss of generality say $a_{11} \neq 0$, then our equation can be rewritten as

$$x_1 = d_1 + \frac{-a_{12}}{a_{11}}x_2 + \cdots + \frac{-a_{1n}}{a_{11}}x_n.$$

Since we can take any values of $x_2, \ldots x_n$ and choose x_1 accordingly to get a solution, and $n \geq 2$, there are infinitely many solutions.

Inductive Step: Assume that a system of m linear equations in more than m variables has either no solutions or infinitely many solutions. Show the same is true for a system of $m+1$ linear equations in more than $m+1$ variables.

Our system is

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = d_1,$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = d_2,$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = d_m,$$

$$a_{(m+1)1}x_1 + a_{(m+1)2}x_2 + \cdots + a_{(m+1)n}x_n = d_{m+1},$$

where $n > m + 1$.

If $a_{(m+1)i} = 0$ for all i, and $d_{m+1} \neq 0$, then there are no solutions.

If $a_{(m+1)i} = 0$ for all i, and $d_{m+1} = 0$, then the last equation is $0 = 0$, and we can eliminate it. This leaves us with m equations in more than m variables, so by inductive hypothesis, there are either no solutions or infinitely many.

Otherwise, some $a_{(m+1)i} \neq 0$; without loss of generality, say $a_{(m+1)n} \neq 0$. Then we can use the last equation to solve for x_n,

$$x_n = b_1x_1 + b_2x_2 + \cdots + b_{n-1}x_{n-1} + d.$$

Substituting this expression for x_n in the first m equations gives us m equations in $n-1$ variables, and since $n > m+1$, we have $n-1 > m$. Therefore, by inductive hypothesis, this system in variables $x_1, x_2, \ldots x_{n-1}$ has either no solutions or infinitely many solutions. If it has none, neither does our original system. If it has infinitely many, each one can be extended to a solution to our original system by setting

$$x_n = b_1x_1 + b_2x_2 + \cdots + b_{n-1}x_{n-1} + d,$$

and so our original system has infinitely many solutions.