Math 24
Spring 2012
Friday, May 25

(1.) TRUE or FALSE?

(a.) Every unitary operator is normal.

(b.) Every orthogonal operator is diagonalizable.

(c.) A matrix is unitary if and only if it is invertible.

(d.) If two matrices are unitarily equivalent, then they are also similar.

(e.) The sum of unitary matrices is unitary.

(f.) The adjoint of a unitary matrix is unitary.

(g.) If T is an orthogonal operator on V,then$[T]_\beta$ is an orthogonal matrix for any ordered basis β for V.

(h.) If all the eigenvalues of a linear operator are 1, then the operator must be unitary or orthogonal.

(i.) A linear operator may preserve the norm, but not the inner product.

(2.) TRUE or FALSE?

(a.) All projections are self-adjoint.

(b.) An orthogonal projection is uniquely determined by its range.

(c.) Every self-adjoint operator is a linear combination of orthogonal projections.

(d.) If T is a projection on W, then $T(x)$ is the vector in W that is closest to x.

(e.) Every orthogonal projection is a unitary operator.

Answers are in the back of the textbook.

(3.) What linear operators T on $\mathbb{R}^2$ are orthogonal?

For a linear operator on R^2 to be orthogonal, its matrix A in the standard basis must be orthogonal; that is, we must have $AA^t = A^tA = I$, or $A^{-1} = A^t$. The textbook tells us that this happens just in case the operator is either a rotation about the origin or a reflection across a line through the origin.

(4.) What linear operators T on $\mathbb{R}^3$ are orthogonal?

For a linear operator on R^3 to be orthogonal, its matrix A in the standard basis must be orthogonal; that is, we must have $AA^t = A^tA = I$, or $A^{-1} = A^t$. We can see that a rotation about a line through the origin or a reflection across a plane containing the origin is orthogonal. So are compositions of such operators.

(5.) Give an example of an orthogonal projection of $\mathbb{R}^2$, and a projection of $\mathbb{R}^2$ that is not orthogonal.

The projection $T(a, b) = (a, 0)$ is orthogonal (it is the orthogonal projection onto the x-axis.) The projection $T(a, b) = (a, a)$ is not orthogonal (it is the vertical projection onto the line $x = y$).

(6.) Show that if V is a finite-dimensional inner product space, a linear operator T is a projection if and only if there is a basis β such that $[T]_\beta$ is a diagonal matrix whose diagonal entries are 0's and 1's, and T is an orthogonal projection if and only if there is an orthonormal basis β with this property.

Suppose T is a projection onto a subspace W. As the textbook explains, this means that $W = R(T)$, for $w \in W$ we have $T(w) = w$, and $V = N(T) \oplus R(T)$. Let α be a basis for $N(T)$ and β be a basis for $R(T)$; then $\alpha \cup \beta$ is a basis for V. For $w \in \alpha$ we have $T(w) = w$ and for $v \in \beta$ we have $T(v) = 0$. Therefore, the matrix of T in the basis $\alpha \cup \beta$ is as required.

Conversely, suppose $\beta = \{v_1, v_2, \ldots, v_n\}$ and $[T]_\beta$ is as described. Then T is the projection onto $span(\{v_i \mid ([T]_\beta)_{ii} = 1\})$ along $span(\{v_i \mid ([T]_\beta)_{ii} = 0\})$, which we can easily check from the definition of projection.

If T is an orthogonal projection, then in the argument above we can take α and β to be orthonormal bases. Because T is orthogonal, $N(T)$ and $R(T)$ are orthogonal complements, and so $\alpha \cup \beta$ is also orthonormal.

For the converse, if $\beta = \{v_1, v_2, \ldots, v_n\}$ is orthonormal, then T is in fact an orthogonal projection; we can check that $N(T)$ and $R(T)$ are orthogonal complements.