Math 24
Spring 2012
Monday, May 21

(1.) TRUE or FALSE? (These questions always deal with finite-dimensional inner product spaces.)

(a.) Every linear operator has an adjoint.
(b.) Every linear operator on V has the form $x \mapsto \langle x, y \rangle$ for some $y \in V$.
(c.) For every linear operator T on V and every ordered basis β for V we have $[T^*]_\beta = [T_\beta]^*$.
(d.) The adjoint of a linear operator is unique.
(e.) For any linear operators T and U and scalars a and b,

$$(aT + bU)^* = aT^* + bU^*.$$

(f.) For any $n \times n$ matrix A, we have $(L_A)^* = L_{A^*}$.
(g.) For any linear operator T we have $(T^*)^* = T$.

Answers are in the back of the textbook.

(2.) Consider the following inconsistent[1] system of linear equations.

$$x_1 + x_2 - x_3 = 0$$

$$2x_1 - x_2 + x_3 = 3$$

$$x_1 - x_2 + x_3 = 2$$

(a.) Find a matrix A and a column vector b such that this system is equivalent to the matrix equation $Ax = b$, where x denotes the column vector $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$.

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 1 \\ 1 & -1 & 1 \end{pmatrix} \qquad b = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}$$

(b.) Find a solution $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ to $Ax = b$ for which $||x||$ is minimal (where $||\ ||$ denotes the standard norm on $\mathbb{R}^3$).

[1] Oops — that word was left over from another problem; this system is consistent.

We want to find a solution $x = A^*u$ in the range of L_{A^*}. That is, we want to solve $AA^*u = b$ for u, and then set $x = A^*u$.

$$AA^*u = b$$

$$\begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 0 & -1 \\ 0 & 6 & 4 \\ -1 & 4 & 3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{3}t \\ -\frac{2}{3}t + \frac{1}{2} \\ t \end{pmatrix}$$

Because we need only one solution, we can set $t = 0$. (Any other value for t will give the same x; you can check this.)

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \end{pmatrix}$$

$$x = A^*u$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

You can check that this is a solution to the original system of equations.

(3.) Let A be the matrix of problem (2), and $T = L_A$.

(a.) What is $T^*(x_1, x_2, x_3)$?

$$A^*x = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 + x_3 \\ x_1 - x_2 - x_3 \\ -x_1 + x_2 + x_3 \end{pmatrix}$$

$$T^*(x_1, x_2, x_3) = (x_1 + 2x_2 + x_3,\ x_1 - x_2 - x_3,\ -x_1 + x_2 + x_3)$$

(b.) Give (specific) geometric descriptions of the subspaces $N(T)$, $R(T)$, $N(T^*)$ and $R(T^*)$, and verify that $R(T^*) = (N(T))^\perp$ and $N(T^*) = (R(T))^\perp$.

A row-reduces to the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$. Since row operations do not change the null space, the solution set to $Ax = 0$ is $x_1 = 0$, $x_2 = x_3$, and a basis for $N(L_A)$ is $\{(0, 1, 1)\}$.

That is, $N(T) = N(L_A)$ is the line through the origin in the direction of $(0, 1, 1)$.

A column-reduces to the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{3} & \frac{2}{3} & 0 \end{pmatrix}$. Since column operations do not change the range, a basis for the range of L_A is $\{(3, 0, -1), (0, 3, 2)\}$.

That is, $R(T) = R(L_A)$ is the plane through the origin orthogonal to the vector $(3, 0, -1) \times (0, 3, 2) = (3, -6, 9)$.

A^* row-reduces to the matrix $\begin{pmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 0 \end{pmatrix}$. Since row operations do not change the null space, the solution set to $Ax = 0$ is $x_1 = \frac{1}{3}x_3$, $x_2 = -\frac{2}{3}x_3$, and a basis for $N(L_{A^*})$ is $\{(1, -2, 3)\}$.

That is, $N(T^*) = N(L_{A^*})$ is the line through the origin in the direction of $(1, -2, 3)$ (or, the direction of $(3, -6, 9)$).

A^* column-reduces to the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix}$. Since column operations do not change the range, a basis for the range of L_* is $\{(1, 0, 0), (0, 1, -1)\}$.

That is, $R(T^*) = R(L_{A^*})$ is the plane through the origin orthogonal to the vector $(1, 0, 0) \times (0, 1, -1) = (0, 1, 1)$.

It is clear from the geometric descriptions that $R(T^*) = (N(T))^\perp$ and $N(T^*) = (R(T))^\perp$.

(4.) Let $T : \mathbb{C}^2 \to \mathbb{C}^2$ be defined by and $T(w, z) = (w + iz,\ w - iz)$.

(a.) What is $T^*(w, z)$?

Let β denote the standard (orthonormal) basis for $\mathbb{C}^2$.

$$[T]_\beta = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \qquad [T^*]_\beta = ([T]_\beta)^* = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$$

$$T^*(w, z) = (w + z, -iw + iz)$$

(b.) Compute the subspaces $N(T)$, $R(T)$, $N(T^*)$ and $R(T^*)$, and verify that $R(T^*) = (N(T))^\perp$ and $N(T^*) = (R(T))^\perp$.

Since these matrices are invertible, $N(T) = N(T^*) = \{0\}$ and $R(T) = R(T^*) = \mathbb{C}^2$.