Math 24
Spring 2012
Monday, May 14
Sample Solutions

(1.) TRUE or FALSE?

(a.) Any linear operator on an n-dimensional vector space that has fewer than n distinct eigenvalues is not diagonalizable.
(b.) Two distinct eigenvalues corresponding to the same eigenvalue are always linearly dependent.
(c.) If λ is an eigenvalue of a linear operator T, then each vector in E_λ is an eigenvalue of T.
(d.) If λ_1 and λ_2 are distinct eigenvalues of a linear operator T, then $E_{\lambda_1} \cap E_{\lambda_2} = \{0\}$.
(e.) Let $A \in M_{n\times n}(F)$ and $\beta = \{v_1, v_2, \ldots, v_n\}$ be an ordered basis for F^n consisting of eigenvectors of A. If Q is the $n \times n$ matrix whose j^{th} column is v_n $(1 \leq j \leq n)$, then $Q^{-1}AQ$ is a diagonal matrix.
(f.) A linear operator T on a finite-dimensional vector space is diagonalizable if and only if the multiplicity of each eigenvalue λ equals the dimension of E_λ.
(g.) Every diagonalizable linear operator on a nonzero vector space has at least one eigenvalue.
(h.) You can always tell from the characteristic polynomial of A whether A is diagonalizable.
(i.) You can sometimes tell from the characteristic polynomial of A whether A is diagonalizable.
(j.) You can always tell from the characteristic polynomial of A whether A is invertible.

Answers are (mostly) in the back of the book. For (h), see the previous homework. For (i), if the characteristic polynomial does not split then A is not diagonalizable, and if the charcteristic polynomial splits and all roots have multiplicity 1, then A is diagonalizable. For (j), A is invertible if and only if its null space is $\{0\}$, that is, if and only if 0 is not an eigenvalue of A, and from the characteristic polynomial you can tell what the eigenvalues are.

(2.) Find an invertible matrix Q and find a diagonalizable matrix B such that either $QAQ^{-1} = B$ or $Q^{-1}AQ = B$. Be sure to say which of these two equations holds for your Q and B.

$$A = \begin{pmatrix} 1 & -7 & 2 \\ 0 & 2 & 0 \\ 0 & -10 & 2 \end{pmatrix}.$$

The characteristic polynomial of A is $(2-\lambda)^2(1-\lambda)$. Eigenvalue $\lambda = 1$ has multiplicity 1 and a basis for the eigenspace is $\{(1,0,0)\}$. Eigenvalue $\lambda = 2$ has multiplicity 2 and a basis for the eigenspace is $\{(2,0,1)\}$. Since this eigenspace does not have dimension 2, A is in fact not diagonalizable.

(3.) For the matrix A in problem (2), find a basis for the eigenspace of A corresponding to each eigenvalue. Describe each of these eigenspaces geometrically. (Be specific. Don't just say "a line"; specify which line.)

The eigenspace E_1 is the x-axis. The eigenspace E_2 is the line in the xz-plane with equations $x = 2z$, $y = 0$.

(4.) Test the matrix A for diagonalizability.

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

The eigenvalues of A are $\lambda = 1$, of multiplicity 2, and $\lambda = 2$, of multiplicity 1. To test for diagonalizability, we must test whether the eigenspace for $\lambda = 1$ has dimension 2. The eigenspace E_λ is the null space of $A - \lambda I$, so we must check the nullity of $A - \lambda I$. By the Dimension Theorem, we can find the nullity of a matrix from its rank.

In our case $\lambda = 1$, $A - \lambda I = A - I = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, and we can see that this matrix has rank 2, so it has nullity 1. Therefore the dimension of the eigenspace is 1 and A is not diagonalizable.

(5.) Suppose a linear operator T on an n-dimensional vector space V has only one eigenvalue $\lambda = 1$, and T is diagonalizable. What can you conclude about T?

What can you say in general about diagonalizable linear operators with a single eigenvalue?

Since T is diagonalizable, V has a basis β consisting of eigenvectors for T. Since the only eigenvalue of T is $\lambda = 1$, every element of β is an eigenvector for that eigenvalue, and so for $v \in \beta$, we have $T(v) = v$. Since T agrees with the identity operator I_V on β, and a linear transformation is determined by its action on a basis, T must be the identity operator: $T = I_V$, and $T(v) = v$ for every $v \in V$.

By similar reasoning, if T is diagonalizable and its only eigenvalue is c, then $T(v) = cv$ for every $v \in V$.

Notice that regardless of our choice of basis α, we have $[T]_\alpha = cI$.

(6.) Show that if T is a diagonalizable linear operator on an n-dimensional vector space V with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$, then each vector v in V can be expressed uniquely as

$$v = v_1 + v_2 + \cdots + v_k$$

where $v_i \in E_{\lambda_i}$.

First, we show that any such expression is unique. Suppose that

$$v_1 + v_2 + \cdots + v_k = w_1 + w_2 + \cdots + w_k,$$

where v_i and w_i are in the eigenspace E_{λ_i}. We must show that $v_i = w_i$ for all i.

We have

$$(v_1 - w_1) + (v_2 - w_2) + \cdots + (v_k - w_k) = 0,$$

where $v_i - w_i$ is in the eigenspace E_{λ_i}. Because eigenvectors corresponding to distinct eigenvalues are linearly independent, the only way this can happen is if we always have $v_i - w_i = 0$, or $v_i = w_i$ for all i.

Now we show that any vector in V can be expressed in this form. Because T is diagonalizable, V has a basis of eigenvectors,

$$\beta = \{v_{1,1}, \ldots, v_{1,m_1}, v_{2,1}, \ldots, v_{2,m_2}, \ldots, v_{k,1}, \ldots, v_{k,m_k}\},$$

where $v_{i,j}$ is an eigenvector for λ_i. Because β is a basis, we can express any v in V as a linear combination of vectors from β,

$$v = a_{1,1}v_{1,1} + \cdots + a_{1,m_1}v_{1,m_1} + a_{2,1}v_{2,1} + \cdots + a_{2,m_2}v_{2,m_2} + \cdots + a_{k,1}v_{k,1} + \cdots + a_{k,m_k}v_{k,m_k}.$$

Grouping together vectors from the same eigenspace, we have

$$v = v_1 + v_2 + \cdots + v_k,$$

where $v_i = a_{i,1}v_{i,1} + \cdots + a_{i,m_i}v_{i,m_i} \in E_{\lambda_i}$.