Math 24
Spring 2012
Wednesday, May 9

1. Show that if T and U are linear transformations from a vector space V to a vector space W, then $R(T+U) \subseteq R(T) + R(U)$.

   Show that if A and B are $m \times n$ matrices, then $rank(A+B) \leq rank(A) + rank(B)$.

   Suppose $v \in R(T+U)$; we must show $v \in R(T) + R(U)$. That is, we must show we can write v as the sum of an element of $R(T)$ and an element of $R(U)$.

   Since v is in the range of $T+U$, by the definition of $T+U$, for some $x \in V$ we have

   $$v = (T+U)(x) = T(x) + U(x).$$

   Since $T(x) \in R(T)$ and $U(x) \in R(U)$, this is what we needed to show.

   For the second part, we must show

   $$dim(R(L_{A+B})) \leq dim(R(L_A)) + dim(R(L_B)).$$

   We know that $L_{A+B} = L_A + L_B$, so we must show

   $$dim(R(L_A + L_B)) \leq dim(R(L_A)) + dim(R(L_B)).$$

   By the first part, $R(L_A + L_B) \subseteq R(L_A) + R(L_B)$, so

   $$dim(R(L_A + L_B)) \leq dim(R(L_A) + R(L_B)),$$

   and we must show that

   $$dim(R(L_A) + R(L_B)) \leq dim(R(L_A)) + dim(R(L_B)).$$

   Let's show that in general $dim(W_1+W_2) \leq dim(W_1)+dim(W_2)$. We know $W_1+W_2 = span(W_1 \cup W_2)$, so if α is a basis for W_1 and β is a basis for W_2, then $\alpha \cup \beta$ generates $W_1 + W_2$. Therefore $dim(W_1 + W_2) \leq size(\alpha) + size(\beta) = dim(W_1) + dim(W_2)$.

2. Suppose that $T : V \to W$ and $U : V \to Z$ are linear transformations between finite-dimensional vector spaces (possibly of different dimensions). When is there a linear transformation $\overline{T} : W \to Z$ such that $U = \overline{T}T$? When is there no such linear transformation?

   You may not be able to find a complete answer; if not, come up with whatever criteria you can. (Examples: Assume U is the zero transformation. Assume U is not the zero transformation, but T is.)

   Can you deduce anything about when a matrix equation $AX = B$ (where X is a matrix of variables) has a solution?

   First, suppose there is such a $\overline{T}$. Then, for $x \in N(T)$, we must have

$$U(x) = \overline{T}T(x) = \overline{T}(T(x)) = \overline{T}(0) = 0,$$

   so also $x \in N(U$. Therefore, if $N(T) \not\subseteq N(U)$, there is no such $\overline{T}$.

   Conversely, we will show that if $N(T) \subseteq N(U)$, then there is such a $\overline{T}$. Let $\{v_1, \ldots, v_k\}$ be a basis for $N(T)$, and extend it to a basis $\{v_1, \ldots v_n\}$ for V. We know that a linear transformation is determined by its action on a basis, so we will have $\overline{T}T = U$ as long as we have $\overline{T}T(v_i) = U(v_i)$ for $i = 1, \ldots, n$.

   For $i \leq k$, we have $v_i \in N(T) \subseteq N(U)$, so $\overline{T}T(v_i) = \overline{T}(T(v_i)) = \overline{T}(0) = 0 = U(v_i)$, however we define $\overline{T}$. For $k < i \leq n$, we must guarantee $\overline{T}T(v_i) = \overline{T}(T(v_i)) = U(v_i)$.

   By the proof of the Dimension Theorem, $\{T(v_{k+1}), \ldots, T(v_n)\}$ is a basis for $R(T)$. Extend it to a basis $\{T(v_{k+1}), \ldots, T(v_n), w_1, \ldots, w_m\}$ for W. We can define $\overline{T}$ on a basis for W any way we want, by a theorem in the textbook, so we can set $\overline{T}(T(v_i)) = U(v_i)$ for $i = k+1, \ldots, n$, and $\overline{T}(w_j) = 0$.

   The answer to the second question is "not really." We already know (by looking at the columns of AX one at at time) that $AX = B$ has a solution if and only if $rank(A|B) = rank(A)$.

   However, we can deduce something about when a matrix equation $XA = B$ has a solution. We are trying to solve $L_X L_A = L_B$ for L_X, like trying to solve $\overline{T}T = U$ for $\overline{T}$, so by the first part we can do this in case $N(L_A) \subseteq N(L_B)$, that is, in case $A\vec{x} = 0 \implies B\vec{x} = 0$.

   By thinking of the rows of A and B as coefficients of linear equations in the systems corresponding to $A\vec{x} = 0$ and $B\vec{x} = 0$, we can see that the equations in $B\vec{x} = 0$ must be linear combinations of the equations in $Ax = 0$; that is, the rows of B must be linear combinations of the rows in A. That is, we must have $rank \begin{pmatrix} A \\ B \end{pmatrix} = rank(A)$.

3. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T(x,y,z) = \left(x,\ \left(\frac{1}{4}x + \frac{3}{4}y - \frac{1}{4}z\right),\ \left(\frac{1}{4}x - \frac{1}{4}y + \frac{3}{4}z\right)\right),$$

α be the standard basis for $\mathbb{R}^3$, and β be the basis $\{(1,1,0),\ (0,1,1),\ (1,0,1)\}$.

(a) Find $[T]_\beta$.

(b) Find $([T]_\beta)^n$. (A^n is just A multiplied by itself n times.)

(c) Find matrices Q and Q^{-1} such that $[T]_\alpha = Q[T]_\beta Q^{-1}$.

(d) Use the fact that $[T]_\alpha = Q[T]_\beta Q^{-1}$ to find $([T]_\alpha)^n$ and $T^n(x,y,z)$. (T^n is just T composed with itself n times.)

(e) Find $\lim_{n\to\infty} ([T]_\beta)^n$ and $\lim_{n\to\infty} ([T]_\alpha)^n$. (The limit of a sequence of matrices is computed entry-by-entry.))

(f) Find $\lim_{n\to\infty} T^n(x,y,z)$.

This problem is a preview of Chapter 5, and is also related to an important application called Markov chains. Suppose (x,y,z) describes the state of some system at a given time (for example, x, y, and z could be the populations of three organisms in an ecosystem, or the net worths of three Monopoly players), and $T(x,y,z)$ always describes the state of the system one "step" later (for example, one fiscal year, or one turn for each player). Then $\lim_{n\to\infty} T^n(x,y,z)$ is the limiting state of the system, the state towards which the system will tend over time, if it starts in state (x,y,z).

$T(1,1,0) = (1,1,0)$, $T(0,1,1) = \left(0, \frac{1}{2}, \frac{1}{2}\right)$, $T(1,0,1) = (1,0,1)$, so $[T]_\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

$([T]_\beta)^n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2^n} & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

$Q = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \qquad Q^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

$([T]_\alpha)^2 = Q[T]_\beta Q^{-1} Q[T]_\beta Q^{-1} = Q([T]_\beta)^2 Q^{-1}$, and in general $[T]_\alpha^n = Q([T]_\beta)^n Q^{-1}$.

Multiplying it out, $[T]_\alpha^n = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} - \frac{1}{2^{n+1}} & \frac{1}{2} + \frac{1}{2^{n+1}} & -\frac{1}{2} + \frac{1}{2^{n+1}} \\ \frac{1}{2} - \frac{1}{2^{n+1}} & -\frac{1}{2} + \frac{1}{2^{n+1}} & \frac{1}{2} + \frac{1}{2^{n+1}} \end{pmatrix}$.

$$T^n(x,y,z) = \left(x,\ \frac{x+y-z}{2} + \frac{-x+y+z}{2^{n+1}},\ \frac{x-y+z}{2} + \frac{-x+y+z}{2^{n+1}}\right).$$

$$\lim_{n\to\infty}([T]_\beta)^n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \lim_{n\to\infty}([T]_\alpha)^n = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

$$\lim_{n\to\infty} T^n(x,y,z) = \left(x,\ \frac{x+y-z}{2},\ \frac{x-y+z}{2}\right).$$