Math 24
Spring 2012
Friday, May 4

(1.) TRUE or FALSE?

(a.) The function $det : M_{n\times n}(F) \to F$ is a linear transformation. (F)
(b.) The determinant of a $n \times n$ matrix is a linear function of each row of the matrix when the other rows are held fixed. (T)
(c.) If $A \in M_{n\times n}(F)$ and $det(A) = 0$ then A is invertible. (F)
(d.) If u and v are vectors in $\mathbb{R}^2$ emanating from the origin, then the area of the parallelogram having u and v as adjacent sides is $det\begin{pmatrix} u \\ v \end{pmatrix}$. (F)
(e.) A coordinate system is right-handed if and only if its orientation equals 1. (T)
(f.) The determinant of a square matrix can be evaluated by cofactor expansion along any row. (T)
(g.) If two rows of a square matrix A are identical, then $det(A) = 0$. (T)
(h.) If B is a matrix obtained from a square matrix A by multiplying a row of A by a scalar, then $det(B) = det(A)$. (F)
(i.) If B is a matrix obtained from a square matrix A by interchanging any two rows, then $det(B) = -det(A)$. (T)
(j.) If B is a matrix obtained from a square matrix A by adding k times row i to row j, then $det(B) = k\, det(A)$. (F)
(k.) If $A \in M_{n\times n}(F)$ has rank n then $det(A) = 0$. (F)
(l.) The determinant of an upper triangular matrix equals the product of its diagonal entries. (T)

(2.) Evaluate the determinant of the following matrix in $M_{3\times 3}(\mathbb{C})$, first by cofactor expansion along any row, second by using elementary row operations to transform it to an upper triangular matrix.

$$\begin{pmatrix} 0 & 1 & 3 \\ -i & 0 & -3 \\ 2 & 3i & 0 \end{pmatrix}$$

Using cofactor expansion along the second row, we get

$$-(-i)\begin{vmatrix} 1 & 3 \\ 3i & 0 \end{vmatrix} + (0)\begin{vmatrix} 0 & 3 \\ 2 & 0 \end{vmatrix} - (-3)\begin{vmatrix} 0 & 1 \\ 2 & 3i \end{vmatrix} = i(-9i) + 3(-2) = 3.$$

Using elementary row operations of type 3, which do not change the determinant, we transform the matrix in the following steps:

$$\begin{pmatrix} 0 & 1 & 3 \\ -i & 0 & -3 \\ 2 & 3i & 0 \end{pmatrix} \begin{pmatrix} -i & 1 & 0 \\ -i & 0 & -3 \\ 2 & 3i & 0 \end{pmatrix} \begin{pmatrix} -i & 1 & 0 \\ 0 & -1 & -3 \\ 2 & 3i & 0 \end{pmatrix} \begin{pmatrix} -i & 1 & 0 \\ 0 & -1 & -3 \\ 0 & i & 0 \end{pmatrix} \begin{pmatrix} -i & 1 & 0 \\ 0 & -1 & -3 \\ 0 & 0 & -3i \end{pmatrix}$$

(Add $R2$ to $R1$; add $(-1)R1$ from $R2$; add $(-2i)R1$ to $R3$; add $(i)R2$ to $R3$.)
The determinant of the resulting matrix is the product of its diagonal entries, $-3i^2 = 3$.

For the remaining problems, let $G : M_{n\times n}(F) \to F$ be any function such that

(a.) G is a linear function of any row, when the other rows are held fixed.
(b.) If two rows of a matrix A are identical, then $G(A) = 0$.
(c.) $G(I_n) = 1$.

Show the following:

First, let's introduce some notation. If we write out a matrix A in terms of its rows, $A = \begin{pmatrix} \vec{r}_1 \\ \vdots \\ \vec{r}_{i-1} \\ \vec{r}_i \\ \vec{r}_{i+1} \\ \vdots \\ \vec{r}_n \end{pmatrix}$, we can define a linear function by $T_{A,i}(\vec{x}) = G\begin{pmatrix} \vec{r}_1 \\ \vdots \\ \vec{r}_{i-1} \\ \vec{x} \\ \vec{r}_{i+1} \\ \vdots \\ \vec{r}_n \end{pmatrix}$. This is in fact a linear function of $\vec{x}$ by condition (a). Notice that $T_{A,i}(\vec{r}_i) = G(A)$. Also notice that if $j \neq i$, then by condition (b), $T_{A,j}(\vec{r}_i) = 0$.

(3.) If B is obtained from A by multiplying row i by the scalar r, then $G(B) = r\,G(A)$. (Hint: Use the fact that G is a linear function of row i when the other rows are held fixed.)

If $A = \begin{pmatrix} \vec{r}_1 \\ \vdots \\ \vec{r}_{i-1} \\ \vec{r}_i \\ \vec{r}_{i+1} \\ \vdots \\ \vec{r}_n \end{pmatrix}$, then $B = \begin{pmatrix} \vec{r}_1 \\ \vdots \\ \vec{r}_{i-1} \\ r(\vec{r}_i) \\ \vec{r}_{i+1} \\ \vdots \\ \vec{r}_n \end{pmatrix}$, and $G(B) = T_{A,i}(r(\vec{r}_i)) = rT_{A,i}(\vec{r}_i) = rG(A)$, by the linearity of $T_{A,i}$.

(4.) If row i of A consists entirely of zeroes, then $G(A) = 0$.

If $\vec{r}_i = \vec{0}$, then $G(A) = T_{A,i}(\vec{r}_i) = T_{A,i}(\vec{0}) = 0$, by the linearity of $T_{A,i}$.

(5.) If B is obtained from A by adding a scalar multiple of row i to row j, then $G(B) = G(A)$. (Hint: Use the fact that G is a linear function of row j when the other rows are held fixed.)

If $A = \begin{pmatrix} \vec{r}_1 \\ \vdots \\ \vec{r}_i \\ \vdots \\ \vec{r}_{j-1} \\ \vec{r}_j \\ \vec{r}_{j+1} \\ \vdots \\ \vec{r}_n \end{pmatrix}$ and $B = \begin{pmatrix} \vec{r}_1 \\ \vdots \\ \vec{r}_i \\ \vdots \\ \vec{r}_{j-1} \\ \vec{r}_j + r(\vec{r}_i) \\ \vec{r}_{j+1} \\ \vdots \\ \vec{r}_n \end{pmatrix}$, then

$G(B) = T_{A,j}(\vec{r}_j + r(\vec{r}_i)) = T_{A,j}(\vec{r}_j) + rT_{A,,j}(\vec{r}_i) = G(A) + 0 = G(A)$.

(6.) If B is obtained from A by interchanging row i and row j, then $det(B) = -det(A)$. (Hint: This type 1 elementary row operation can be accomplished by a combination of type 2 and type 3 operations.)

$$\begin{pmatrix} \vec{r}_i \\ \vec{r}_j \end{pmatrix} \begin{pmatrix} \vec{r}_i + \vec{r}_j \\ \vec{r}_j \end{pmatrix} \begin{pmatrix} \vec{r}_i + \vec{r}_j \\ -\vec{r}_i \end{pmatrix} \begin{pmatrix} \vec{r}_j \\ -\vec{r}_i \end{pmatrix} \begin{pmatrix} \vec{r}_j \\ \vec{r}_i \end{pmatrix}$$

This sequence shows that rows i and j can be interchanged by adding row j to row i; subtracting row i from row j; adding row j to row i; multiplying row j by -1. By the previous problems, the net result is to multiply the value of G by -1.

(7.) $G(A) = det(A)$ for any $n \times n$ matrix A. (Hint: A can be transformed by elementary row operations to either I_n or a matrix with a row of zeroes.)

Since elementary row operations are invertible, A can be obtained from B via elementary row operations, where either $B = I_n$ or B has a row of zeroes. By condition (c) and problem (4), $G(B) = det(B)$. By problems (3), (5), and (6), elementary row operations affect the value of G and the value of the determinant in the same way. Therefore $G(A) = det(A)$.