Math 24
Spring 2012
Monday, April 16

(1.) TRUE or FALSE? In each part, V, W, and Z denote finite-dimensional vector spaces with ordered bases α, β and γ respectively, $T : V \to W$ and $U : W \to Z$ denote linear transformations, and A and B denote matrices.

(a.) $[UT]_\alpha^\gamma = [T]_\alpha^\beta [U]_\beta^\gamma$. (F)

(b.) $[T(v)]_\beta = [T]_\alpha^\beta [v]_\alpha$ for all $v \in V$. (T)

(c.) $[U(w)]_\beta = [U]_\alpha^\beta [w]_\beta$ for all $w \in W$. (F)

(d.) $[I_V]_\alpha = I$. (T)

(e.) $[T^2]_\alpha^\beta = ([T]_\alpha^\beta)^2$. (F)

(f.) $A^2 = I$ implies that $A = I$ or $A = -I$. (F)

(g.) $T = L_A$ for some matrix A. (F)

(h.) $A^2 = 0$ implies that $A = 0$, where 0 denotes the zero matrix. (F)

(i.) $L_{A+B} = L_A + L_B$. (T)

(j.) If A is square and $A_{ij} = \delta_{ij}$ for all i and j, then $A = I$. (T)

(2.) If $A = \begin{pmatrix} 2 & 5 \\ -3 & 1 \\ 4 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & -2 & 0 \\ 1 & -1 & 4 \\ 5 & 5 & 3 \end{pmatrix}$, which of the matrix products AB and BA is defined?

Find the second column of that matrix product.

A is 3×2 and B is 3×3, so BA is defined, and is 3×2.

The entries of the second column of BA are the dot products of the rows of B with the second column of A. This gives $\begin{pmatrix} 13 \\ 12 \\ 36 \end{pmatrix}$.

(3.) Write down a matrix A such that $A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3x - 2y + z \\ x - 2z \end{pmatrix}$.

$$A = \begin{pmatrix} 3 & -2 & 1 \\ 1 & 0 & -2 \end{pmatrix}$$

(4.) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that rotates counterclockwise around the origin by ninety degrees (so if v is on the positive x axis, then $T(v)$ is on the positive y-axis), and $U : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that projects every point perpendicularly onto the x-axis. Let β be the standard basis for $\mathbb{R}^2$.

(a.) Find explicit expressions for $T(x, y)$ and $U(x, y)$. Use these to write explicit expressions for $UT(x, y)$ and $TU(x, y)$. (Recall that $UT(x, y)$ denotes $U(T(x, y))$.) Find $UT(1, 0)$, $UT(0, 1)$, $TU(1, 0)$ and $TU(0, 1)$. Use these values to write down the matrices $[UT]_\beta$ and $[TU]_\beta$.

$T(x, y) = (-y, x) \quad U(x, y) = (x, 0)$
$UT(x, y) = U(-y, x) = (-y, 0) \quad TU(x, y) = T(x, 0) = (0, x)$
$UT(1, 0) = (0, 0) \quad UT(0, 1) = (-1, 0)$
$TU(1, 0) = (0, 1) \quad TU(0, 1) = (0, 0)$
The columns of $[UT]_\beta$ are the β-coordinates of $UT(1, 0)$ and $UT(0, 1)$:
$[UT]_\beta = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \quad [TU]_\beta = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

(b.) Find $U(1, 0)$, $U(0, 1)$, $T(1, 0)$ and $T(0, 1)$. Use these values to write down the matrices $[U]_\beta$ and $[T]_\beta$.

$U(1, 0) = (1, 0) \quad U(0, 1) = (0, 0)$
$T(1, 0) = (0, 1) \quad T(0, 1) = (-1, 0)$
$[U]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad [T]_\beta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

(c.) Use matrix multiplication to compute $[U]_\beta[T]_\beta$ and $[T]_\beta[U]_\beta$. Compare with your answers to part (a); did you get what you should?

Yes, $[U]_\beta[T]_\beta = [UT]_\beta$ and $[T]_\beta[U]_\beta = [TU]_\beta$.

(d.) Compute the matrix product $[U]_\beta[T]_\beta \begin{pmatrix} x \\ y \end{pmatrix}$. This should give you $[UT(x, y)]_\beta$. Compare this with your answer to part (a); did you get what you should?

Yes, $[U]_\beta[T]_\beta \begin{pmatrix} x \\ y \end{pmatrix} = [UT(x, y)]_\beta$.

(5.) Find a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ other than $I_{\mathbb{R}^2}$ and $-I_{\mathbb{R}^2}$, with the property that $T(T(v)) = v$ for every v in $\mathbb{R}^2$. Use T to find a matrix A such that $A \neq I$ and $A \neq -I$ but $A^2 = I$. (Recall that $I_{\mathbb{R}^2}$ denotes the identity transformation on $\mathbb{R}^2$, so $I_{\mathbb{R}^2}(v) = v$ and $-I_{\mathbb{R}^2}(v) = -v$. Your function T should be different from either of these.)

$T(x, y) = (y, x)$ works. $A = [T]_\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, if β is the standard basis for $\mathbb{R}^2$.

So does $U(x, y) = (x, -y)$. $[U]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

(6.) Suppose that n teams are playing in a large and somewhat random tournament, in which no two teams play each other more than once, but some pairs of teams may not play each other at all. You would like to have some way of comparing two teams who did not play each other.

Let A be the $n \times n$ matrix where entry A_{ij} is 1 if teams i and j play each other, and 0 if they do not. Prove that $(A^2)_{ij} = 0$ if and only if there is no team that played both team i and team j during the tournament. You might want to remember that $(A^2)_{ij} = A_{i1}A_{1j} + A_{i2}A_{2j} + \cdots + A_{1n}A_{nj}$, and ask yourself when $A_{i2}A_{2j}$ is equal to 0, and when it is equal to 1.

How do you interpret $(A^2)_{ij}$ in the case where $(A^2)_{ij} \neq 0$?

$A_{ik}A_{kj}$ is 1 if team i played team k and team k played team j, and 0 otherwise. Thus $(A^2)_{ij} = A_{i1}A_{1j} + A_{i2}A_{2j} + \cdots + A_{1n}A_{nj}$ is the number of teams that played both team i and team j.

Now define a new $n \times n$ matrix B where B_{ij} is 1 if team i defeated team j, and 0 if not (either teams i and j did not play each other, or team j defeated team i). Assume there are no ties. If $(A^2)_{ij} = 6$, $(B^2)_{ij} = 4$, and $(B^2)_{ji} = 1$, what (if anything) can you conclude about how teams i and j performed against teams they both played?

$B_{ik}B_{kj}$ is 1 if team i defeated team k and team k defeated team j, and 0 otherwise. Thus $(B^2)_{ij} = B_{i1}B_{1j} + B_{i2}B_{2j} + \cdots + B_{1n}B_{nj}$ is the number of teams that played both team i and team j.

The values given indicate that of the 6 teams that played both team i and team j, there were 4 teams that were defeated by team i and in turn defeated team j, there was 1 team that was defeated by team j and in turn defeated team i, and the remaining team either defeated both teams or was defeated by both teams.

It looks to me like team i is the stronger team.