Math 24 Spring 2006 Assignment 1 Key

Due Friday, April 7.

(1) Is $\ln x$ (natural logarithm) a linear transformation on the vector space $\mathbb{R}$? Justify your answer.

Answer: No, it is not. A linear transformation must commute with the vector operations of addition and scalar multiplication, and it is not the case that $\ln(x+y) = \ln x + \ln y$ or that $\ln(cx) = c \ln x$.

(2) Define the operation $*$ on $\mathbb{R}^{>0}$ (positive real numbers) by

$$a * b = \sqrt{ab}.$$

Is $(\mathbb{R}^{>0}, *)$ a group? If not, which of the group axioms fails?

Answer: $(\mathbb{R}^{>0}, *)$ is not a group. There is no identity element, because for a given a the element b which gives $a * b = a$ is a itself, which means there is no single element which works for all a. Therefore there also cannot be inverses. In fact, this operation is not even associative: the only time $a * (b * c) = (a * b) * c$ is when $a = c$.

[all three axiom failures not required; one is sufficient.]

(3) Define the operation $*$ on $2\mathbb{Z} = \{2n : n \in \mathbb{Z}\}$ by

$$a * b = a + b.$$

Is $(2\mathbb{Z}, *)$ a group? If not, which of the group axioms fails?

Answer: $(2\mathbb{Z}, *)$ is a group.

[unneeded but probably commonly-given information: the operation is closed because the sum of two even numbers is even. It is associative because it is just ordinary addition. The identity is $0 = 2 \cdot 0$ and the inverse of $2n$ is $-2n = 2(-n)$.]

(4) Consider the field $(\{0, 1, 2\}, + \bmod 3, \cdot \bmod 3)$ similar to one discussed in class. Show that for all a, b in the field,

$$(a + b)^3 = a^3 + b^3.$$

Answer: If we expand the binomial, we get

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

Since we are working mod 3, the middle two terms are equal to zero no matter what a and b are, so the right hand side simplifies to $a^3 + b^3$.