Fall 2014 MATH 24 Linear Algebra
Final Exam

Name:____

1 2 3 4 5 6 7 8 9 10

Instructions:

(1) Exam time: 3 hours.
(2) There are 10 problems in this exam. Only 8 problems you choose (circle) will be graded and they are graded 5 points each.
(3) All notations are standard. Unless specified, the field of scalars is always $\mathbb{R}$.
(4) Problems are *not* ordered by their difficulty.
(5) You do not have to show every step of your calculation; for example, you do not need to show the intermediate steps for finding a reduced row echelon form, or for finding the inverse of a matrix.
(6) However, you need to explain your work to receive full credit. The correct answer alone does not guarantee full credit.

Problem 1. *Find all possible values of* (a, b) *for which the following matrix has eigenvalue* 2.

$$\begin{pmatrix} a & b & a+b \\ 2 & 2 & 3 \\ b & 0 & a \end{pmatrix}$$

Problem 2. *Let* $\beta = \{v_1, v_2, v_3\}$ *be a basis in* $\mathbb{R}^3$. *Let*

$$u_1 = 2v_1 + 2v_2 + v_3, \quad u_2 = 2v_2 + 3v_3, \quad u_3 = 2v_1 + 3v_2.$$

Show that $\gamma = \{u_1, u_2, u_3\}$ *is also a basis, and find all vectors* v *in* $\mathbb{R}^3$ *whose* β*-coordinate agrees with* γ*-coordinate, i.e.,* $[v]_\beta = [v]_\gamma$.

Problem 3. *Given a system of linear equations $Ax = b$ where $rank(A) = 3$. We know that v_1, v_2 and v_3 are three solutions to the system where:*

$$v_1 = (1, 2, 3, 4)^t, \quad v_2 + v_3 = (5, 6, 7, 8)^t.$$

The system may have many solutions. Find the solution which is closest to the point $(-4, 0, 1, 0)^t$ with respect to the standard Euclidean distance.

Problem 4. *Let T be a rotation in $\mathbb{R}^3$ around the axis $(1,1,0)^t$ by 120° counter-clockwise (by right-hand-rule). Find the matrix representation of T with respect to the standard basis $\{e_1, e_2, e_3\}$.*

Problem 5. *Find all possible values of a for which the following matrix is invertible.*

$$\begin{pmatrix} 0 & 2 & 0 & 4 \\ a-1 & 3a+2 & 3 & 3-a \\ 0 & 0 & 0 & 3 \\ 5 & 4a-1 & a+1 & 2a-1 \end{pmatrix}$$

Problem 6. *Find all possible* (a, b) *such that the following two matrices are similar to each other.*

$$A = \begin{pmatrix} b & 0 & 2 \\ 0 & 2 & 3 \\ 1 & 0 & b \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & a \end{pmatrix}$$

Problem 7. *In the space* $\mathbb{R}^6$*, let:*

$$\alpha_1 = (1,1,1,1,1,1)^t, \quad \alpha_2 = (1,-1,1,-1,1,-1)^t, \quad \alpha_3 = (1,0,0,1,1,0)^t,$$
$$\beta_1 = (1,2,3,1,2,3)^t, \quad \beta_2 = (1,1,2,2,1,1)^t, \quad \beta_3 = (2,1,4,2,3,2)^t.$$

Let $V_1 = span(\alpha_1, \alpha_2, \alpha_3)$ *and* $V_2 = span(\beta_1, \beta_2, \beta_3)$*. Find bases for* $V_1 + V_2$ *and* $V_1 \cap V_2$ *respectively.*

Problem 8. *Let T be the reflection on $\mathbb{R}^3$ across the plane $3x + 4y + 5z = 0$, i.e., every vector is transformed to its mirror image with respect to the "mirror" $3x + 4y + 5z = 0$. Let A be the matrix representing T with respect to the standard basis. Find* $\det(A)$. *In addition, find an orthonormal basis in $\mathbb{R}^3$ such that T has a diagonal matrix representation.*

Problem 9. *Find a matrix X such that*

$$X^2 = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

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Problem 10. *Find the least square approximation of the form* $y = ax^2 + b$ *for the data points* $(-1, 1)$, $(0, 0)$, $(1, 1)$ *and* $(2, 3)$.

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