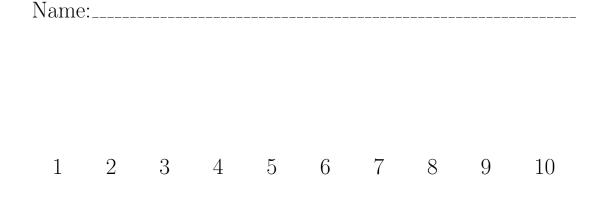
Fall 2014 MATH 24 Linear Algebra Final Exam



Instructions:

- (1) Exam time: 3 hours.
- (2) There are 10 problems in this exam. Only 8 problems you choose (circle) will be graded and they are graded 5 points each.
- (3) All notations are standard. Unless specified, the field of scalars is always \mathbb{R} .
- (4) Problems are *not* ordered by their difficulty.
- (5) You do not have to show every step of your calculation; for example, you do not need to show the intermediate steps for finding a reduced row echelon form, or for finding the inverse of a matrix.
- (6) However, you need to explain your work to receive full credit. The correct answer alone does not guarantee full credit.

Problem 1. Find all possible values of (a, b) for which the following matrix has eigenvalue 2.

$$\begin{pmatrix} a & b & a+b \\ 2 & 2 & 3 \\ b & 0 & a \end{pmatrix}$$

Problem 2. Let $\beta = \{v_1, v_2, v_3\}$ be a basis in \mathbb{R}^3 . Let

 $u_1 = 2v_1 + 2v_2 + v_3, \quad u_2 = 2v_2 + 3v_3, \quad u_3 = 2v_1 + 3v_2.$

Show that $\gamma = \{u_1, u_2, u_3\}$ is also a basis, and find all vectors v in \mathbb{R}^3 whose β coordinate agrees with γ -coordinate, i.e., $[v]_{\beta} = [v]_{\gamma}$.

Problem 3. Given a system of linear equations Ax = b where rank(A) = 3. We know that v_1 , v_2 and v_3 are three solutions to the system where:

$$v_1 = (1, 2, 3, 4)^t, \quad v_2 + v_3 = (5, 6, 7, 8)^t.$$

The system may have many solutions. Find the solution which is closest to the point $(-4, 0, 1, 0)^t$ with respect to the standard Euclidean distance.

Problem 4. Let T be a rotation in \mathbb{R}^3 around the axis $(1,1,0)^t$ by 120° counterclockwise (by right-hand-rule). Find the matrix representation of T with respect to the standard basis $\{e_1, e_2, e_3\}$. **Problem 5.** Find all possible values of a for which the following matrix is invertible.

$$\begin{pmatrix} 0 & 2 & 0 & 4 \\ a-1 & 3a+2 & 3 & 3-a \\ 0 & 0 & 0 & 3 \\ 5 & 4a-1 & a+1 & 2a-1 \end{pmatrix}$$

Problem 6. Find all possible (a, b) such that the following two matrices are similar to each other. $(b \ 0 \ 2)$ $(2 \ 0 \ 0)$

$$A = \begin{pmatrix} b & 0 & 2 \\ 0 & 2 & 3 \\ 1 & 0 & b \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & a \end{pmatrix}$$

Problem 7. In the space \mathbb{R}^6 , let:

 $\alpha_1 = (1, 1, 1, 1, 1, 1)^t, \quad \alpha_2 = (1, -1, 1, -1, 1, -1)^t, \quad \alpha_3 = (1, 0, 0, 1, 1, 0)^t, \\ \beta_1 = (1, 2, 3, 1, 2, 3)^t, \quad \beta_2 = (1, 1, 2, 2, 1, 1)^t, \quad \beta_3 = (2, 1, 4, 2, 3, 2)^t.$

Let $V_1 = span(\alpha_1, \alpha_2, \alpha_3)$ and $V_2 = span(\beta_1, \beta_2, \beta_3)$. Find bases for $V_1 + V_2$ and $V_1 \cap V_2$ respectively.

Problem 8. Let T be the reflection on \mathbb{R}^3 across the plane 3x + 4y + 5z = 0, i.e., every vector is transformed to its mirror image with respect to the "mirror" 3x + 4y + 5z = 0. Let A be the matrix representing T with respect to the standard basis. Find det(A). In addition, find an orthonormal basis in \mathbb{R}^3 such that T has a diagonal matrix representation.

Problem 9. Find a matrix X such that

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$$X^2 = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

Problem 10. Find the least square approximation of the form $y = ax^2 + b$ for the data points (-1, 1), (0, 0), (1, 1) and (2, 3).