$\boxed{30}$ 1. A tank contains 100 gal of water and 50 oz of a chemical. Water containing a concentration of $\frac{1}{4}(1+\frac{t}{2})$ oz/gal of this chemical flows into the tank at a rate of 2 gal/min, and the mixture flows out at the same rate.

(a) Write a differential equation for the amount of chemical in the tank at any time.

(b) Find the amount of chemical in the tank at any time.

a) $\frac{dQ}{dt}$ = rate in − rate out

rate in: $2\,\text{gal/min} \cdot \frac{1}{4}(1+\frac{t}{2})\,\text{oz/gal.} = \frac{1}{2}(1+\frac{t}{2})\,\text{oz/min.}$

rate out: $2\,\text{gal/min.} \cdot \frac{Q}{100}\,\text{oz/gal.} = \frac{Q}{50}\,\text{oz/min.}$

$$\boxed{\frac{dQ}{dt} = \frac{1}{2}(1+\frac{t}{2}) - \frac{Q}{50}}$$

b) $\frac{dQ}{dt} + \frac{Q}{50} = \frac{1}{2} + \frac{t}{4}$

integrating factor $\mu(t) = e^{\int p(t)} = e^{t/50}$

$e^{t/50}\frac{dQ}{dt} + e^{t/50}\frac{Q}{50} = \frac{1}{2}e^{t/50} + \frac{1}{4}te^{t/50}$

$Qe^{t/50} = 25e^{t/50} + \frac{1}{4}(50te^{t/50} - 50^2e^{t/50}) + c$

$Qe^{t/50} = 25e^{t/50} + \frac{25}{2}te^{t/50} - 25^2e^{t/50} + c$

$Q = 25 + \frac{25}{2}t - 625 + ce^{-t/50}$

integration by parts

$u = t \qquad v = 50e^{t/50}$

$du = 1 \qquad dv = e^{t/50}$

$\int te^{t/50} = 50te^{t/50} - \int 50e^{t/50}$

$= 50te^{t/50} - 50^2e^{t/50}$

$Q(0) = 50$

$50 = 25 - 625 + c$

$c = 650$

$Q = 25 + \frac{25}{2}t - 625 + 650e^{-t/50}$

$$\boxed{Q = \frac{25}{2}t - 600 + 650e^{-t/50}}$$

$\boxed{20}$ 2. Find the solution of the initial value problem.

$$y' + 2y = te^{-2t}, \quad y(1) = 0$$

$\mu(t) = e^{\int 2\,dt} = e^{2t}$ is the integrating factor:

$$\underbrace{e^{2t}y' + 2e^{2t}y}_{} = e^{2t} \cdot te^{-2t} = t$$

$$\frac{d}{dt}(e^{2t}y) = t$$

$$e^{2t}y = \int t\,dt = \frac{1}{2}t^2 + C$$

$$y = \frac{1}{2}t^2e^{-2t} + Ce^{-2t}$$

$$0 = y(1) = \frac{1}{2}e^{-2} + Ce^{-2}$$

so $C = -\frac{1}{2}$

thus $y = \frac{1}{2}(t^2 - 1)e^{-2t}$

15 3. Find the general solution to:

$$\frac{dy}{dx} = \frac{x^2}{y},$$

$y \frac{dy}{dx} = x^2$

$\int y \, dy = \int x^2 \, dx$

$\frac{1}{2} y^2 = \frac{1}{3} x^3 + C$

$y^2 = \frac{2}{3} x^3 + C$

$y = \pm \sqrt{\frac{2}{3} x^3 + C}$

15 4. *WITHOUT FINDING A SOLUTION* determine an interval in which the solution of the initial value problem is guaranteed to exist.

$$(4-t^2)y' + 2ty = 3t^2, \quad y(1) = -3$$

In standard form this is

$$y' + \underbrace{\frac{2t}{4-t^2}}_{p(t)} y = \underbrace{\frac{3t^2}{4-t^2}}_{g(t)}$$

$p(t)$ and $g(t)$ are both continuous except when $4-t^2=0$, i.e. $t=\pm 2$

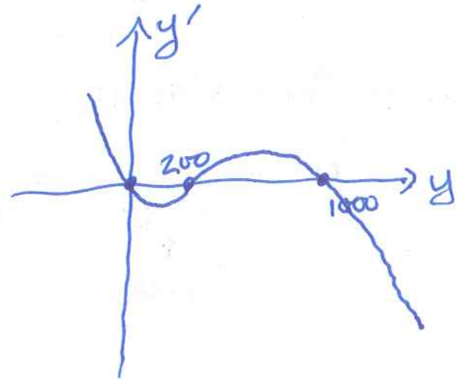
The largest interval that avoids these discontinuities and includes the t-value of the initial condition $(t=1)$ is $(-2, 2)$.

35 5. Suppose a population y is modelled by the equation

$$y' = -y\left(1-\frac{y}{a}\right)\left(1-\frac{y}{1000}\right)$$

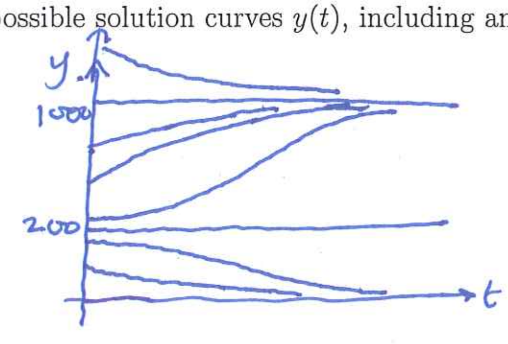
(a) For $a = 200$, sketch:

- the graph of y' as a function of y



- the phase line

- several possible solution curves $y(t)$, including any equilibrium solutions.

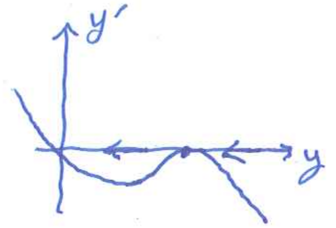


(b) For arbitrary $a > 0$, characterise the stability of the equilibrium solutions. *Do not assume $a < 1000$.*

Cases: $a < 1000$: As in (a), the equilibria 0 and 1000 are stable whereas a is unstable

$a > 1000$: By symmetry of y', now 0 and a are stable whereas 1000 is unstable

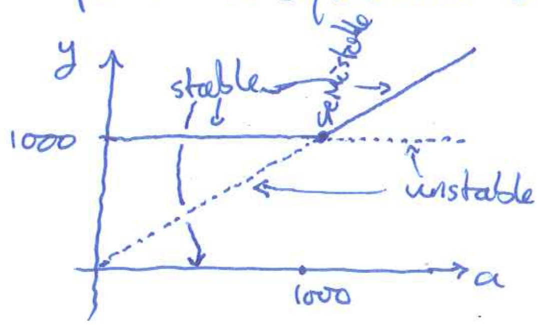
$a = 1000$:



0 is stable while 1000 is semistable

(c) Sketch a bifurcation diagram for the parameter a.

This plots the equilibrium solutions as a function of a



$\boxed{30}$ 6. Find the general solution to the following differential equations. *You do not have to justify that your solution is the general solution.*

(a) $y'' - 6y' + 18y = 0$

$r^2 - 6r + 18 = 0$

$r = \frac{6 \pm \sqrt{36 - 4 \cdot 18}}{2} = 3 \pm \frac{\sqrt{-36}}{2} = 3 \pm 3i$

$\boxed{y = c_1 e^{3t} \sin(3t) + c_2 e^{3t} \cos(3t)}$

(b) $4y'' - 4y' + 3y = 0$

$4r^2 - 4r + 3 = 0$

$r = \frac{4 \pm \sqrt{16 - 48}}{8} = \frac{1}{2} \pm \frac{\sqrt{-32}}{8} = \frac{1}{2} \pm i\,\frac{\sqrt{2}}{2}$

$\boxed{y = c_1 e^{t/2} \sin\left(\frac{\sqrt{2}}{2}t\right) + c_2 e^{t/2} \cos\left(\frac{\sqrt{2}}{2}t\right)}$

$\boxed{25}$ 7. (a) Find two constants n such that $y = t^n$ is a solution to the differential equation

$$t^2y'' + 3ty' - 3y = 0$$

$$y' = nt^{n-1}$$
$$y'' = n(n-1)t^{n-2}$$
$$[n(n-1) + 3n - 3]t^n = 0$$
$$n^2 + 2n - 3 = 0$$
$$(n+3)(n-1) = 0$$
$$n = 1, -3$$

(b) Write down the general solution to the differential equation for $t < 0$ and use the Wronskian to justify that this is the general solution.

$$y = c_1 t + c_2 t^{-3}$$

$$W[t, t^{-3}] = \begin{vmatrix} t & t^{-3} \\ 1 & -3t^{-4} \end{vmatrix} = -4t^{-3}$$

This is nonzero for $t<0$, hence $\{t, t^{-3}\}$ form a fundamental set of solutions.

$\boxed{30}$ 8. Solve the initial value problem using the method of undetermined coefficients.

$$y'' - 2y' + y = 3te^{2t}, \quad y(0) = 2, \quad y'(0) = 4$$

$y'' - 2y' + y = 0$

$r^2 - 2r + 1 = (r-1)^2 = 0$, $r = 1$

$y_c = c_1 e^t + c_2 t e^t$

Let $Y = Ate^{2t} + Be^{2t}$

$Y' = 2Ate^{2t} + Ae^{2t} + 2Be^{2t} = 2Ate^{2t} + (A+2B)e^{2t}$

$Y'' = 4Ate^{2t} + 2Ae^{2t} + (2A + 4B)e^{2t}$

$= 4Ate^{2t} + (4A + 4B)e^{2t}$

plug in:

$4Ate^{2t} + (4A+4B)e^{2t} - 2(2Ate^{2t} + (A+2B)e^{2t}) + Ate^{2t} + Be^{2t} = 3te^{2t}$

collect terms:

$(4A - 4A + A)te^{2t} + (4A + 4B - 2A - 4B + B)e^{2t} = 3te^{2t}$

$Ate^{2t} + (2A+B)e^{2t} = 3te^{2t}$

$A = 3$, $2A + B = 0 \Rightarrow B = -6$

$Y = 3te^{2t} - 6e^{2t}$

$y = c_1 e^t + c_2 te^t + 3te^{2t} - 6e^{2t}$

$y' = c_1 e^t + c_2 e^t + c_2 te^t + 3e^{2t} + 6te^{2t} - 12e^{2t}$

plug in initial conditions:

$2 = c_1 - 6$, $c_1 = 8$

$4 = c_1 + c_2 + 3 - 12 \Rightarrow 4 = 8 + c_2 - 9 \Rightarrow c_2 = 5$

$$\boxed{y = 8e^t + 5te^t + 3te^{2t} - 6e^{2t}}$$