3.5 + 4.3

$$\underbrace{c_n y^{(n)} + \dots + c_1 y' + c_0 y}_{\text{degree } n \text{ with constant coefficients}} = g(t) \quad (*)$$

$g(t)$ – RHS of (*)	$y_p(t)$ – a particular solution (*)
$a_n t^n + \dots + a_1 t + a_0$	$t^s (A_n t^n + \dots + A_0)$ $s=$ number of times <u>zero</u> is the root of $c_n x^n + \dots + c_1 x + c_0 = 0$ the characteristic eqn of (*)
$e^{\alpha t}(a_n t^n + \dots + a_0)$	$t^s e^{\alpha t}(A_n t^n + \dots + A_0)$ $s=$ # of times α is a root of the characteristic eqn $c_n x^n + \dots + c_0 = 0$
$e^{\alpha t}\cos\beta t\,(a_n t^n + \dots + a_0)$ $+$ $e^{\alpha t}\sin\beta t\,(b_n t^n + \dots + b_0)$ we can have $a_i = 0$ or $b_i = 0$!	$t^s e^{\alpha t}\cos\beta t\,(A_n t^n + \dots + A_0)$ $+$ $t^s e^{\alpha t}\sin\beta t\,(B_n t^n + \dots + A_0)$ $s=$ # of times $\alpha + i\beta$ is a root of the characteristic equation $c_n x^n + \dots + c_0 = 0$

Example 1

$$Y'' + Y' + Y = 0$$

$\downarrow$

characteristic eqn

$$x^2 + x + 1 = 0 \longrightarrow \text{roots} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

general solution is

$$y(t) = C_1 e^{-\frac{1}{2}t} \sin\left(\frac{\sqrt{3}}{2} t\right) + C_2 e^{-\frac{1}{2}t} \cos\left(\frac{\sqrt{3}}{2} t\right)$$

Non homogeneous cases

A $\quad Y'' + Y' + Y = \underbrace{t^3 + 1}_{\text{polynomial of degree 3}}$

$\underset{\text{zero}}{0}$ is <u>not</u> a root of $x^2 + x + 1 = 0 \Rightarrow \boxed{s = 0}$

$Y_p(t)$ Should be searched in the form

$$Y_p = A_3 t^3 + A_2 t^2 + A_1 t + A_0$$

$$Y_p' = 3 A_3 t^2 + 2 A_2 t + A_1$$

$$Y_p'' = 6 A_3 t + 2 A_2$$

$$(6 A_3 t + 2 A_2)$$
$$+$$
$$(3 A_3 t^2 + 2 A_2 t + A_1)$$
$$+$$
$$(A_3 t^3 + A_2 t^2 + A_1 t + A_0)$$
$$\|$$
$$t^3 + 1$$

$$\begin{array}{ccccc}
1.t^3 & & A_3 t^3 & & \\
+ & & & & \\
0.t^2 & & 3A_3t^2 & +A_2t^2 & \\
+ & = & & & \\
0.t & & 6A_3t & +2A_2t & +A_1t \\
+ & & & & \\
1 & & 2A_2 & +A_1 & +A_0
\end{array}$$

$\rightsquigarrow$ $A_3 = 1$

$A_2 = -3$

$A_1 = 0$

$A_0 = 7$

(B) $y'' + y' + y = e^{-t/2} \underbrace{(t^2+3)}_{\text{Polynomial of degree two}}$

$-\frac{1}{2}$ is <u>not</u> a root of the characteristic eqn

$x^2 + x + 1 = 0 \Rightarrow \boxed{s = 0}$

$y_p(t) = e^{-t/2}(A_2t^2 + A_1t + A_0) \rightsquigarrow$ $A_0 = 4/9$, $A_1 = 0$, Algebra $A_2 = 12/9$

(c) 4

$$y'' + y' + y = \cos(\sqrt{3}\, t)\, \underbrace{t^2}_{\text{polynomial of degree two}}$$

$0 + \sqrt{3}i$ is not a root of $x^2 + x + 1 = 0$

$\Rightarrow \boxed{s = 0}$

$$y_p(t) = e^{0t}\cos(\sqrt{3}t)(A_2 t^2 + A_1 t + A_0) + e^{0t}\sin(\sqrt{3}t)(B_2 t^2 + B_1 t + B_0)$$

6x6 system!

$A_2 = \frac{1}{343}\sqrt{3}\, 49$ $\qquad B_2 = \frac{-2}{343} 49$

$A_1 = \frac{1}{343}\sqrt{3}\, 84$ $\qquad B_1 = \frac{2}{343} \cdot 161$

$A_0 = \frac{1}{343}\sqrt{3}\,(-222)$ $\qquad B_0 = \frac{-18}{343}$

D

$$y'' + y' + y = e^{-t/2} \cos\left(\frac{\sqrt{3}}{2} t\right) \underbrace{(t+1)}_{\substack{\text{Polynomial} \\ \text{of} \\ \text{degree } 1}}$$

$-\frac{1}{2} + \frac{\sqrt{3}}{2} i$ is A root of

$x^2 + x + 1 = 0$

A simple root $\Rightarrow$ $\boxed{s = 1}$

$$y_p(t) = e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}}{2} t\right) t \left(A_1 t + A_0\right)$$

$$+$$

$$e^{-t/2} \sin\left(\frac{\sqrt{3}}{2} t\right) t \left(B_1 t + B_0\right)$$

Example ②

$$y'' + 4y' + 3y = 0$$

$x^2 + 4x + 3 = 0 \rightsquigarrow x = -3$

$x = -1$

$$y(t) = c_1 e^{-3t} + c_2 e^{-t}$$

Ⓐ $y'' + 4y' + 3y = e^{-3t}(t+3)$

$r = -3$ is a root! a simple root $\Rightarrow \boxed{S = 1}$

$$y_p(t) = e^{-3t} t^1 (A_1 t + A_0)$$

Ⓑ $y'' + 4y' + 3y = e^{4t}(t+3)$

4 is not a root! $\Rightarrow S = 0$

$$y_p(t) = e^{4t}(A_1 t + A_0)$$

Ⓒ $y'' + 4y' + 3y = \cos 2t = e^{0t}\cos(2t)$

$0 + 2i$ is ~~a root~~ not a root $\Rightarrow \boxed{S = 0}$

$$y_p(t) = \cos(2t) A_0 + \sin(2t) B_0$$

Example 3

$$y'' - 6y' + 9y = 0$$

$$\downarrow$$

$$x^2 - 6x + 9 = 0 \Rightarrow x = 3 \quad \text{a double root!}$$

$$y(t) = c_1 e^{3t} + c_2 t e^{3t}$$

A $\quad y'' - 6y' + 9y = e^{3t}(t+2)$

3 is a double root of $x^2 - 6x + 9$!

$\Rightarrow \boxed{s = 2}$

$$y_p(t) = e^{3t} t^2 (A_1 t + A_0)$$

B $\quad y'' - 6y' + 9y = e^{3t} \cos t \,(t+2)$

$3+i$ is <u>not</u> a root of $x^2 - 6x + 9$

$\Rightarrow s = 0$

$$y_p(t) = e^{3t} \cos t \,(A_1 t + A_0) + e^{3t} \sin t \,(B_1 t + B_0)$$

(C) $Y'' - 6Y' + 9Y = t+1$

0 is not a root of $x^2 - 6x + 9 \Rightarrow \boxed{s=0}$

$Y_p(t) = A_1 t + A_0$

(D) $Y'' - 6Y' + 9Y = 3\cos(2t)$

$0+2i$ is not a root $\Rightarrow \boxed{s=0}$

$Y_p(t) = A\cos(2t) + B\sin(2t)$

$Y_p'(t) = -2A\sin(2t) + 2B\cos(2t)$

$Y_p''(t) = -4A\cos(2t) - 4B\sin(2t)$

$Y_p'' - 6Y_p' + 9Y_p = 3\cos(2t)$

$(-4A\cos(2t) - 4B\sin(2t)) - 6(-2A\sin(2t) + 2B\cos(2t))$

$+9(A\cos(2t) + B\sin(2t)) = 3\cos(2t)$

$\Leftrightarrow$

$$\begin{matrix} \cos(2t)(-4A - 6B + 9A) & & 3\cos(2t) \\ + & = & + \\ \sin(2t)(-4B + 12A + 9B) & & 0\sin(2t) \end{matrix}$$

$$\begin{cases} 5A - 12B = 3 & \times 5 \\ & \text{and add} \\ 12A + 5B = 0 & \times 12 \end{cases}$$

$$\Rightarrow \begin{cases} 25A + 144A = 15 \\ 169A = 15 \end{cases} \Rightarrow \begin{matrix} A = 15/169 \\ B = -36/169 \end{matrix}$$

Thus the general equation is

$$y(t) = c_1 t e^{3t} + c_2 e^{3t} t + \frac{15}{169}\cos(2t) - \frac{36}{169}\sin(2t)$$

Example IVP
$$\begin{cases} y'' - 6y' + 9y = 3\cos 2t \\ y(0) = \frac{15}{169} \\ y'(0) = -\frac{108}{169} \end{cases}$$

$$\Rightarrow C_2 = \frac{-36}{169} \qquad C_1 = 0$$

$$y(t) = -\frac{36}{169} t e^{3t} + \frac{15}{169}\cos 3t - \frac{36}{169}\sin 3t$$

Example

Find the general solution of

$$y'' - 6y' + 9y = t + 1 + 3\cos 2t$$

Thm

If $\tilde{Y}_1$ is a solution of

$$y'' + p(t)y' + q(t)y = g_1(t)$$

and $\tilde{Y}_2$ a solution of

$$y'' + p(t)y' + q(t)y = g_2(t)$$

then $\tilde{Y}_1 + \tilde{Y}_2$ is a solution of

$$y'' + p(t)y' + q(t)y = g_1(t) + g_2(t)$$

example we Apply the thm:

$$y'' - 6y' + 9y = t + 1 = g_1(t)$$

$$y'' - 6y' + 9y = 3\cos 2t = g_2(t)$$

$$\tilde{Y}_2 = \frac{15}{169}\cos 2t - \frac{36}{169}\sin 2t \qquad \tilde{Y}_1 = \frac{1}{9}t + \frac{5}{27}$$

the general solution is

$$Y(t) = c_1 e^{3t} + c_2 t e^{3t} + \underbrace{\frac{1}{9}t + \frac{5}{27}}_{\tilde{Y}_1}$$

$$+ \underbrace{\frac{15}{169}\cos 2t - \frac{36}{169}\sin 2t}_{\tilde{Y}_2}$$