Systems of First Order ODEs, Part II

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Outline

Outline

The Set Up

Consider the general equation

$$\mathbf{x}' = \mathbf{A}\mathbf{x},$$

where **A** is a constant matrix.

- equilibrium solutions correspond to solutions of $\mathbf{A}\mathbf{x} = 0$.
- $\det \mathbf{A} \neq 0$ if and only if $\mathbf{x}(t) = 0$ is the only equilibrium solution.
- When $\det \mathbf{A} \neq 0$ it is interesting to see whether solutions approach or diverge from the equilibrium $\mathbf{x}(t) = 0$ as $t \to \pm\infty$.

The Set Up

Let **A** be a 2×2 matrix

- a solution $\mathbf{x}(t) = (x_1(t), x_2(t))$ of

$$\mathbf{x}' = \mathbf{A}\mathbf{x},$$

  is a curve in the x_1x_2-plane.
- $\mathbf{x}'(t) = (x_1'(t), x_2'(t))$ is the velocity vector.
- Plotting **Ax** gives us the phase plane.

The Strategy

Let $\mathbf{A}$ be a 2×2 matrix (with $\det(\mathbf{A}) \neq 0$).

- Suppose $\mathbf{x}(t) = \zeta e^{rt}$ solves
$$\mathbf{x}' = \mathbf{A}\mathbf{x},$$
where $\zeta = (\zeta_1, \ldots, \zeta_n)^t$.
- $\mathbf{x}(t) = \zeta e^{rt}$ is a solution if and only if $\mathbf{A}\zeta = r\zeta$.

Moral

Solving $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is equivalent to finding eigenvalues and eigenvectors of $\mathbf{A}$.

Outline

Example 1: Saddle

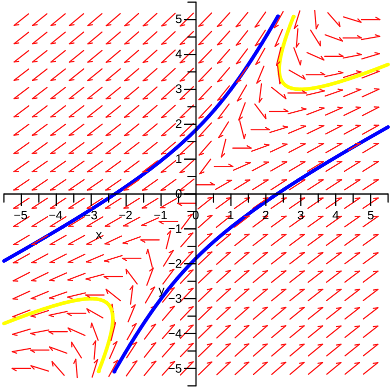
Let $\mathbf{A} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$, then solve

$$\mathbf{x}' = \mathbf{A}\mathbf{x}.$$

- Step 1: Assume solution looks like $\mathbf{x}(t) = \zeta e^{rt}$.
- Step 2: Find eigenvalues of **A**.
- Step 3: Find the corresponding eigenvectors.
- Step 4: Find the General Solution

The equilibrium solution $\mathbf{x}(t) = (0, 0)$ is a saddle.

Example 1: Saddle



Example 2: Sink

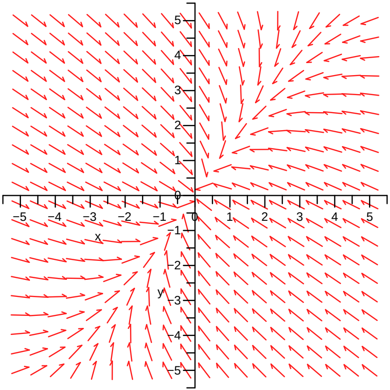
Let $\mathbf{A} = \begin{pmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix}$, then solve

$$\mathbf{x}' = \mathbf{A}\mathbf{x}.$$

- Step 1: Assume solution looks like $\mathbf{x}(t) = \zeta e^{rt}$.
- Step 2: Find eigenvalues of $\mathbf{A}$.
- Step 3: Find the corresponding eigenvectors.
- Step 4: Find the General Solution

The equilibrium solution $\mathbf{x}(t) = (0, 0)$ is a sink.

Example 2: Sink



Example 3: Source

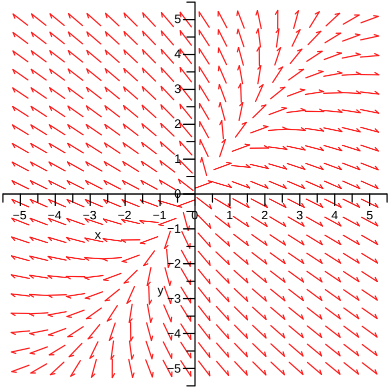
Let $\mathbf{A} = \begin{pmatrix} 3 & -\sqrt{2} \\ -\sqrt{2} & 2 \end{pmatrix}$, then solve

$$\mathbf{x}' = \mathbf{A}\mathbf{x}.$$

- Step 1: Assume solution looks like $\mathbf{x}(t) = \zeta e^{rt}$.
- Step 2: Find eigenvalues of $\mathbf{A}$.
- Step 3: Find the corresponding eigenvectors.
- Step 4: Find the General Solution

The equilibrium solution $\mathbf{x}(t) = (0, 0)$ is a source.

Example 3: Source



Equilibrium Points

Consider a 2×2 constant coefficient system with two nonzero real, distinct eigenvalues λ_1, λ_2.

- If $\lambda_1 < 0 < \lambda_2$, then the origin is a saddle.
- If $\lambda_1 < \lambda_2 < 0$, then the origin is a sink. All solutions tend to $(0,0)$ as $t \to \infty$, and most tend towards $(0,0)$ in the direction of the λ_2-eigenvector. Why?
- If $0 < \lambda_2 < \lambda_1$, then $(0,0)$ is a source. All solutions (except the equil. sol.) go to infinity as $t \to \infty$, and most solution curves *leave* the origin in the direction of the λ_2-eigenvector. Why?

Equilibrium Points

Question

What happens if $\lambda_1 = 0$ and $\lambda_2 \neq 0$?

Outline

Useful Facts

Theorem

Let $\mathbf{A}$ *be a* 2×2 *matrix with* **real** *entries and let* $\phi(t)$ *be a complex valued solution to*

$$\mathbf{x}' = \mathbf{A}\mathbf{x},$$

then

1. $\overline{\phi(t)}$ *is also a solution*
2. *and, therefore,* $\text{Re}(\phi(t))$ *and* $\text{Im}(\phi(t))$ *are also solutions.*

Useful Facts

Theorem

Let **A** *be a* 2×2 *matrix with* **real** *entries and and suppose* $\lambda = \mu + i\nu$ $(\nu \neq 0)$ *is an eigenvalue of* **A** *with a corresponding eigenvector* λ*. Then* $\overline{\lambda} = \mu - i\nu$ *is an eigenvalue of* **A** *and* $\overline{\zeta}$ *is a corresponding eigenvector.*

Useful Facts

Theorem

Let $\mathbf{A}$ *be a* 2×2 *real matrix with complex eigenvalues* $\lambda_1 = \mu + i\nu$ *and* $\lambda_2 = \mu - i\nu$*, where* $\nu \neq 0$*. Let* ζ *be an eigenvector. Then*

1. $\zeta = V + i\,W$*, where* $V, W \in \mathbb{R}^2$ *are non-zero.*
2. $V = \mathrm{Re}(\zeta)$ *and* $W = \mathrm{Im}(\zeta)$ *are linearly independent vectors. In particular,* $V, W \neq \mathbf{0}$*.*

Why?

A Strategy

Suppose **A** has complex eigenvalues $\lambda = \mu + i\nu$, $\nu \neq 0$. To solve $\mathbf{x}' = \mathbf{A}\mathbf{x}$ we:

1. Assume $\mathbf{x}(t) = e^{\lambda t}\zeta$
2. Find eigenvector $\zeta = \mathbf{V} + i\,\mathbf{W}$ associated to λ
3. $\mathbf{x}(t) = e^{\lambda t}\zeta$ is a complex valued solution.
4. Then $\{\mathrm{Re}(\mathbf{x}(t)), \mathrm{Im}(\mathbf{x}(t))\}$ forms a fundamental set of real-valued solutions. **Why?**

Example I: Real Part Non-Zero

Let $\mathbf{A} = \begin{pmatrix} -2 & -3 \\ 3 & -2 \end{pmatrix}$, then solve

$$\mathbf{x}' = \mathbf{A}\mathbf{x}.$$

- Step 1: Assume solution looks like

$$\mathbf{x}(t) = \zeta e^{rt}.$$

- Step 2: Find eigenvalues of $\mathbf{A}$

$$\lambda_1 = -2 + 3i, \lambda_2 = -2 - 3i$$

- Step 3: Find the corresponding eigenvectors.

$$\zeta_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}, \zeta_2 = \begin{pmatrix} 1 \\ +i \end{pmatrix}$$

Example I: Real Part Non-Zero

- Step 4: A non-trivial complex solution is given

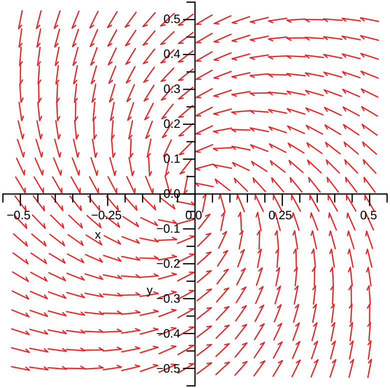
$$\phi(t) = e^{\lambda_1 t}\zeta_1 = e^{-2t}\begin{pmatrix} \cos(3t) + i\sin(3t) \\ \sin(3t) - i\cos(3t) \end{pmatrix}$$

- Step 5: The general solution is given by

$$\begin{aligned} \mathbf{x}(t) &= c_1\,\mathrm{Re}(\phi(t)) + c_2\,\mathrm{Im}(\phi(t)) \\ &= c_1 e^{-2t}\begin{pmatrix} \cos(3t) \\ \sin(3t) \end{pmatrix} + c_2 e^{-2t}\begin{pmatrix} \sin(3t) \\ -\cos(3t) \end{pmatrix} \end{aligned}$$

The equilibrium solution $\mathbf{x}(t) = (0, 0)$ is a spiral point.

Example I: Real Part Non-Zero



Example II: Purely Imaginary Eigenvalues

Let $\mathbf{B} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$, then solve $\mathbf{x}' = \mathbf{B}\mathbf{x}$.

- Step 1: Assume solution looks like

$$\mathbf{x}(t) = \zeta e^{rt}.$$

- Step 2: Find eigenvalues of **B**

$$\lambda_1 = 2i,\ \lambda_2 = -2i.$$

- Step 3: Find the corresponding eigenvectors

$$\zeta_1 = \begin{pmatrix} 1 \\ +i \end{pmatrix},\ \zeta_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

Example II: Purely Imaginary Eigenvalues

- Step 4: Find Complex Solutions

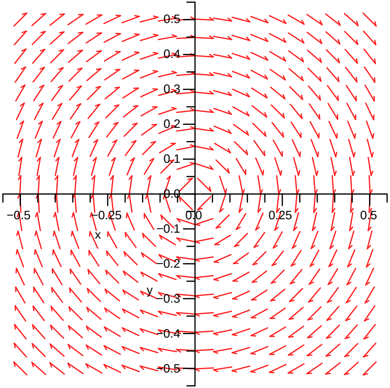
$$\phi(t) = e^{\lambda_1 t}\zeta_1 = \begin{pmatrix} \cos(2t) + i\sin(2t) \\ -\sin(2t) + i\cos(2t) \end{pmatrix}$$

- Step 5: The General Solution is given by

$$\begin{aligned} \mathbf{x}(t) &= c_1 \operatorname{Re}(\phi(t)) + c_2 \operatorname{Im}(\phi(t)) \\ &= c_1 \begin{pmatrix} \cos(2t) \\ -\sin(2t) \end{pmatrix} + c_2 \begin{pmatrix} \sin(2t) \\ \cos(2t) \end{pmatrix} \end{aligned}$$

The equilibrium solution $\mathbf{x}(t) = (0, 0)$ is a center.

Example II: Purely Imaginary Eigenvalues



Equilibrium Points

Consider a 2×2 constant coefficient system with complex eigenvalues $\lambda = \alpha \pm i\beta$ ($\beta \neq 0$).

- If $\alpha < 0$, the solutions spiral towards the origin as $t \to \infty$ and we say $(0, 0)$ is a spiral sink.
- If $\alpha > 0$, the solutions spiral away from the origin as $t \to \infty$ and we say $(0, 0)$ is a spiral source.
- If $\alpha = 0$, the solutions are periodic and we say $(0, 0)$ is a center.

Outline

The Set Up

Let **A** be a 2×2 real matrix with one repeated real eigenvalue λ. There are two cases.

1. λ has two linearly independent eigenvectors ζ_1 and ζ_2.
2. λ has only one linearly indep. eigenvector ζ.

The Set Up

Case 1: λ has two linearly independent eigenvectors ζ_1 and ζ_2.

- General solution is $\phi(t) = c_1 e^{\lambda t}\zeta_1 + c_2 e^{\lambda t}\zeta_2$.
- $\lambda > 0$ then origin is a source. $\lambda < 0$ origin is a sink.

Case 2: λ has one linearly independent eigenvector ζ.

- $e^{\lambda t}\zeta$ is a non-trivial solution.
- How do you get a fundamental set of solutions? That is, how do we get the general solution? The key will be the existence of a vector η such that

$$(\mathbf{A} - \lambda I)\eta = \zeta.$$

Example: Repeated Eigenvalue

Let $\mathbf{A} = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}$, then solve $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

- There is one eigenvalue of **A**: $\lambda = -2$
- $\text{mult}_{\text{geom}}(-2) = 1$. In particular, the eigenvectors associated to $\lambda = -2$ are all scalar multiples of

$$\zeta = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

- $\mathbf{x}_1(t) = e^{\lambda t}\zeta = e^{-2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is a non-trivial solution.

Example: Repeated Eigenvalue

The system $\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix} \mathbf{x}$ implies

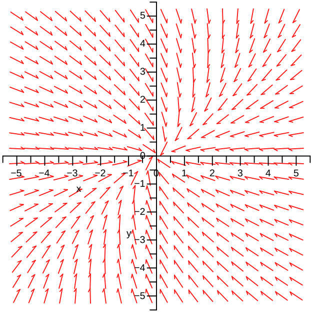
- $\frac{dy}{dt} = -2y$, hence $y(t) = y_0 e^{-2t}$
- Then $\frac{dx}{dt} = -2x + y_0 e^{-2t}$
- So $x(t) = y_0 t e^{-2t} + x_0 e^{-2t}$.
- Hence, the general solution is given by

$$\phi(t) = x_0 e^{-2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y_0 \left(e^{-2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + t e^{-2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)$$

- Notice that $\eta = (0, 1)^t$ satisfies:

$$(\mathbf{A} - (-2)I)\eta = \zeta.$$

Example: Repeated Eigenvalue



Useful Facts

Theorem

Let $\mathbf{A}$ *be a* 2×2 *real matrix with one repeated real eigenvalue* λ*. And suppose the* λ*-eigenvectors are of the form* $c\zeta$ *for any* $c \neq 0 \in \mathbb{R}$ *(i.e.,* $\text{mult}_{\text{geom}}(\lambda) = 1$*), then there exists a non-zero vector* η *such that*

$$(\mathbf{A} - \lambda\mathbf{I})\eta = \zeta.$$

- It then follows that

$$\mathbf{x}_1(t) = e^{\lambda t}\zeta \text{ and } \mathbf{x}_2(t) = e^{\lambda t}\eta + te^{\lambda t}\zeta$$

form a fundamental set of solutions to the system $\mathbf{x}' = \mathbf{Ax}$.

The Theorem

Theorem

Suppose

$$\mathbf{x}' = \mathbf{A}\mathbf{x}$$

is a const. coeff. 2×2 system where $\mathbf{A}$ has a repeated eigenvalue λ with geometric multiplicity 1. Then the general solution has the form

$$c_1 e^{\lambda t}\zeta + c_2(e^{\lambda t}\eta + te^{\lambda t}\zeta)$$

where ζ is a λ-eigenvector and η satisfies $(\mathbf{A} - \lambda I)\eta = \zeta$.

Equivalently...

The Theorem (restated)

Theorem

Suppose

$$\mathbf{x}' = \mathbf{A}\mathbf{x}$$

is a const. coeff. 2×2 *system where* $\mathbf{A}$ *has a repeated eigenvalue* λ *with geometric multiplicity* 1*. Then the general solution has the form*

$$e^{\lambda t}\hat{\eta} + te^{\lambda t}\hat{\zeta}$$

where $\hat{\eta} = (x_0, y_0)$ *is an arbitrary initial condition and* $\hat{\zeta} = (\mathbf{A} - \lambda\mathbf{I})\hat{\eta}$*. If* $\hat{\zeta}$ *is zero, then* $\hat{\eta}$ *is an eigenvector. Otherwise,* $\hat{\zeta}$ *is an eigenvector.*

Exercises

Find the general solution to the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where

1. $\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$
2. $\mathbf{A} = \begin{pmatrix} 4 & -1 \\ 1 & 2 \end{pmatrix}$