Series Solutions of Second Order Linear ODEs

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Outline

Outline

The Definition

Definition

The expression

$$\sum_{j=0}^{\infty} a_j,$$

where the a_j's are real (or complex numbers) is called a **series**. For each $N = 1, 2, 3, \ldots$ the expression

$$S_N = \sum_{j=0}^{N} a_j = a_0 + a_1 + \cdots + a_N$$

is called the N-th partial sum of the series.

Convergence of a Series

Definition

The series $\sum_{j=0}^{\infty} a_j$, is said to converge if

$$\lim_{N \to \infty} S_N = \lim_{N \to \infty} \sum_{j=0}^{N} a_j$$

exists. Otherwise we say the series **diverges**.

A Convergent Series

The series $\sum_{j=0}^{\infty} \frac{1}{2^j}$ converges to 2:

- $S_n = 1 + \frac{1}{2} + \frac{1}{4} + \cdots \frac{1}{2^n}$
- $S_n + \frac{1}{2^{n+1}} = S_{n+1} = 1 + \frac{1}{2} S_n$
- Solving for S_n we get

$$S_n = \frac{1 - \frac{1}{2^{n+1}}}{1 - \frac{1}{2}} = 2\left(1 - \frac{1}{2^{n+1}}\right)$$

- Therefore,

$$\lim_{n \to \infty} S_n = 2.$$

A Divergent Series

The series $\sum_{j=1}^{\infty} \frac{1}{j}$ diverges:

- $S_1 = 1$
- $S_2 = \frac{3}{2}$
- $S_4 = 1 + \frac{1}{2} + (\frac{1}{3} + \frac{1}{4}) \geq \frac{4}{2}$
- In general
$$S_{2^k} \geq \frac{k+2}{2}.$$
- Therefore
$$\lim_{n\to\infty} S_n = \infty$$

Convergence Tests: The Comparison Test

Theorem

Suppose that $\sum_{j=0}^{\infty} a_j$ is a convergent series where $a_j \geq 0$ for all j. If $\{b_j\}_{j=1}^{\infty}$ is a sequence of numbers such that $|b_j| \leq a_j$ for all j, then the series $\sum_{j=0}^{\infty} b_j$ converges.

Convergence Tests: The Comparison Test

The series $\sum_{j=0}^{\infty} \frac{\sin(j)}{2^j}$ converges:

- We recall that $\sum_{j=0}^{\infty} \frac{1}{2^j} = 2$.
- $|\frac{\sin(j)}{2^j}| \leq |\frac{1}{2^j}|$ for all j.
- Hence by the Comparison Test the series

$$\sum_{j=0}^{\infty} \frac{\sin(j)}{2^j}$$

converges.

Convergence Tests: The Ratio Test

Theorem

Consider a series $\sum_{j=0}^{\infty} a_j$ of non-zero terms. If

$$\lim_{j\to\infty} \frac{|a_{j+1}|}{|a_j|} < 1,$$

then the series converges.

Convergence Tests: The Ratio Test

The series $\sum_{j=1}^{\infty} \frac{2^j}{j!}$ converges:

- $a_j = \frac{2^j}{j!}$
- $\lim_{j\to\infty} \frac{|a_{j+1}|}{|a_j|} = \frac{2}{j+1} = 0$
- Therefore by the Ratio Test, the series converges.

Convergence Tests: The Alternating Series Test

Theorem

Let $\{b_j\}_{j=1}^{\infty}$ *be a sequence of nonnegative numbers such that*

1. $b_1 \geq b_2 \geq b_3 \geq \cdots \geq 0;$
2. $\lim_{j\to\infty} b_j = 0.$

Then the series

$$\sum_{j=1}^{\infty}(-1)^j b_j$$

converges

Convergence Tests: The Alternating Series Test

The series $\sum_{j=1}^{\infty} \frac{(-1)^j}{j}$ converges:

- let $b_j = \frac{1}{j}$
- then $b_j \geq b_{j+1} \geq 0$ and $\lim_{j\to\infty} b_j = 0$
- Therefore by the Alternating Series Test the series converges.

The Definition

Definition

The expression

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n$$

is said to be a power series expanded about x_0. For each $N = 1, 2, 3, \cdots$ the expression

$$S_N(x) = \sum_{j=0}^{N} a_j (x - x_0)^j$$

is said to be the N-th partial sum of the Power series.

The Definition

Definition

The power series $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ is said to

1. converge at x if

$$\lim_{N\to\infty} S_N(x) = \lim_{N\to\infty} \sum_{j=0}^{N} a_j(x - x_0)^j$$

exists.

2. converge absolutely at x if the series $\sum_{n=0}^{\infty} |a_n||x - x_0|^n$ converges at x; that is, $\lim_{N\to\infty} \sum_{n=0}^{N} |a_n||x - x_0|^n$ exists.

Absolute Convergence implies Convergence

Proposition

If the series $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ converges absolutely at x, then it converges. The converse need not be true.

Example

The power series $\sum_{j=0}^{\infty} \frac{(-1)^j}{j} x^j$ converges at $x = 1$ (by the alternating series test), but it does not converge absolutely at $x = 1$ since the harmonic series

$$\sum_{j=1}^{\infty} \frac{1}{j}$$

diverges.

Interval of Convergence

Proposition

Assume $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges for $x = c$. Then the power series converges for all x such that

$$|x - x_0| < r = |c - x_0|.$$

Hence, the set

$$\{x \in \mathbb{R} : \sum_{n=0}^{\infty} a_n(x-x_0)^n \text{ converges } \}$$

is an interval centered at x_0.

Radius of Convergence

Definition

The radius of convergence ρ of the power series $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ is

$$\rho = \text{Max}\{r : \sum_{n=0}^{\infty} a_n(x - x_0)^n \text{ converges for all } |x - x_0| < r\}.$$

Real Analytic

Definition

A function $f : U \subset \mathbb{R} \to \mathbb{R}$ is said to be real analytic if for each $x_0 \in U$ $f(x)$ may be represented by a convergent power series on an interval $I \subset U$ of positive radius centered at x_0:

$$f(x) = \sum a_n (x - x_0)^n.$$

Properties

Let $f(x) = \sum a_n(x - x_0)^n$ and $g(x) = \sum b_n(x - x_0)^n$ be power series centered at x_0 which converge on intervals I_1 and I_2 containing x_0 (resp.). Then on $I_1 \cap I_2$ we have

1. $f(x) \pm g(x) = \sum(a_n \pm b_n)(x - x_0)^n$
2. $f(x)g(x) = \sum_{m=0}^{\infty} \sum_{j+k=m}(a_j b_k)(x - x_0)^m$.

The Ratio Test

Theorem (Ratio Test)

Consider the power series $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ *and assume* $\lim_{j\to\infty} |\frac{a_{j+1}}{a_j}|$ *exists and equals L. Then the power series*

1. *converges for x such that* $|x - x_0|L < 1$,
2. *diverges for x such that* $|x - x_0|L > 1$, *and*
3. *for x such that* $|x - x_0|L = 1$ *we don't know.*

The Ratio Test: Examples

1. $cos(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} x^{2j}$ converges for all x.
2. $\sum_{n=1}^{\infty} \frac{n^2}{2^n}(x-3)^n$ converges for all x such that $|x-3| < 2$ and diverges for $|x-3| > 2$. Need to check by hand the case $|x-3| = 2$
3. $\sum_{n=1}^{\infty} \frac{(x+1)^n}{n2^n}$ converges absolutely for $|x+1| < 2$ and diverges for $|x+1| > 2$. Need to check the case $|x+1| = 2$ by hand.

Shifting Indices

1. Consider the series $\sum_{n=k}^{\infty} a_n x^n$
2. Make the substitution $m = n - k$
3. Then

$$\sum_{n=k}^{\infty} a_n x^n = \sum_{m=0}^{\infty} a_{m+k} x^{m+k}$$
$$= \sum_{n=0}^{\infty} a_{n+2} x^{n+2}$$

Shifting Indices: Examples

Write the following series so that the generic term involves x^n

1. $\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$.
2. $\sum_{n=1}^{\infty} n a_n x^{n-1} + x\sum_{n=0}^{\infty} a_n x^n$.
3. $\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n$.

Differentiating and Integrating

Definition

Let $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ be a power series.

1. The derived series is $\sum_{n=1}^{\infty} na_n(x - x_0)^{n-1}$
2. The integrated series is $\sum_{n=0}^{\infty} a_n \frac{(x-x_0)^{n+1}}{n+1}$

Theorem

The derived and integrated series have the same radius of convergence as $\sum_{n=0}^{\infty} a_n(x - x_0)^n$.

Infinitely Differentiable

Theorem

Let $f(x)$ be a real analytic function defined on an open interval I. Then f is continuous and has continuous, real analytic derivatives of all orders. In fact, the derivatives of f are obtained by differentiating its series representation term by term.

Infinitely Differentiable

Corollary

Let f be represented by a convergent power series on an interval of positive radius centered at x_0

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

then

$$a_n = \frac{f^{(n)}(x_0)}{n!}.$$

Examples

1. Find the radius of convergence of the following power series.
   a) $\sum_{n=0}^{\infty} \frac{n}{2^n} x^n$
   b) $\sum_{n=0}^{\infty} \frac{(2x+1)^n}{n^2}$
2. Find the Taylor Series of $f(x) = \frac{1}{1-x}$ at $x_0 = 0$.

Outline

Example

1. Consider the differential equation $y'' + y = 0$.
2. Assume that $y(x) = \sum_{n=0}^{\infty} a_n x^n$.
3. We obtain the recurrence relation

$$a_{2k} = (-1)^k \frac{a_0}{(2k)!} \text{ and } a_{2k+1} = (-1)^k \frac{a_1}{(2k+1)!}.$$

4. Then

$$\begin{aligned} y(x) &= a_0 \sum_k (-1)^k \frac{x^{2k}}{(2k)!} + a_1 \sum_k^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \\ &= a_0 \cos(x) + a_1 \sin(x) \end{aligned}$$

The Method

1. Consider $P(x)y'' + Q(x)y' + R(x)y = 0$
(where P, Q, R are polynomials with no common factors.)
2. Suppose $P(x_0) \neq 0$, then x_0 is called an ordinary point. Otherwise we say x_0 is singular.
3. Then on some interval I containing x_0 we can write the ODE as
$$y'' + p(x)y' + q(x)y = 0.$$
4. Assume $y(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$ and converges for $|x - x_0| < \rho$.
5. Substitute y, y' and y'' into ODE and try to find a recurrence relation for the a_n's. (This will require us to write the rational functions p and q as power series centered at x_0.)

Airy's Equation: Series Solution at $x_0 = 0$

Find a power series solution to $y'' - xy = 0$ in a neighborhood of $x = 0$.

1. Assume $y(x) = \sum_{n=0}^{\infty} a_n x^n$
2. $y'' = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$.
3. Then since $y'' - xy = 0$ we get

$$
\begin{aligned}
0 &= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - x\sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty} a_n x^{n+1} \\
&= 2a_2 + \sum_{n=1}^{\infty}((n+2)(n+1)a_{n+2} - a_{n-1})x^n
\end{aligned}
$$

Airy's Equation: Series Solution at $x_0 = 0$

We then conclude

- $a_2 = 0$
- we have the general recurrence relation

$$a_{n+2} = \frac{a_{n-1}}{(n+2)(n+1)} \quad n \geq 1.$$

- Which implies that for $n \geq 1$

$$\begin{aligned} a_{3n} &= \frac{a_0}{(2 \cdot 3)(5 \cdot 6) \cdots (3n-1)(3n)} \\ a_{3n+1} &= \frac{a_1}{(3 \cdot 4)(6 \cdot 7) \cdots (3n)(3n+1)} \\ a_{3n+2} &= a_2 = 0 \end{aligned}$$

Airy's Equation: Series Solution at $x_0 = 0$

It then follows that our solution $y(x)$ has a Taylor series expansion of the form:

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} a_{3n} x^{3n} + \sum_{n=0}^{\infty} a_{3n+1} x^{3n+1} + \sum_{n=0}^{\infty} a_{3n+2} x^{3n+2} \\ &= \sum_{n=0}^{\infty} a_{3n} x^{3n} + \sum_{n=0}^{\infty} a_{3n+1} x^{3n+1} \\ &= a_0 \left\{ 1 + \sum_{n=1}^{\infty} \frac{1}{(2\cdot 3)(5\cdot 6)\cdots(3n-1)(3n)} x^{3n} \right\} \\ &\quad + a_1 \left\{ x + \sum_{n=1}^{\infty} \frac{1}{(3\cdot 4)(6\cdot 7)\cdots(3n)(3n+1)} x^{3n+1} \right\} \end{aligned}$$

Airy's Equation: Series Solution at $x_0 = 0$

Setting

- $y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{1}{(2\cdot3)(5\cdot6)\cdots(3n-1)(3n)} x^{3n}$
- $y_2(x) = x + \sum_{n=1}^{\infty} \frac{1}{(3\cdot4)(6\cdot7)\cdots(3n)(3n+1)} x^{3n+1}$

We can conclude that y_1 and y_2 are analytic functions which

1. Have an infinite radius of convergence (why?)
2. Solve Airy's Equation on $-\infty < x < \infty$ (why?)
3. Form a fundamental set of solutions on $-\infty < x < \infty$ (why?). Hence, the general solution to Airy's equation is

$$y(x) = c_1 y_1(x) + c_2 y_2(x).$$

Airy's Equation: Series Solution at $x_0 = 1$

Find a power series solution to $y'' - xy = 0$ in a neighborhood of $x = 1$.

1. Assume $y(x) = \sum a_n(x-1)^n$
2. $y'' = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n$.
3. Then since $y'' - xy = 0$ we have

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n - x\sum_{n=0}^{\infty}a_n(x-1)^n \\ &= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n \\ &\quad -(1+(x-1))\sum_{n=0}^{\infty}a_n(x-1)^n \end{aligned}$$

Airy's Equation: Series Solution at $x_0 = 1$

Continuing we have

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n \\ &\quad - \left\{ a_0 + \sum_{n=1}^{\infty}(a_n + a_{n-1})(x-1)^n \right\} \\ &= (2a_2 - a_0) + \sum_{n=1}^{\infty}((n+2)(n+1)a_{n+2} - a_n - a_{n-1})(x-1)^n \end{aligned}$$

Airy's Equation: Series Solution at $x_0 = 1$

From which we deduce:

- $a_2 = \frac{a_0}{2}$
- $a_{n+2} = \frac{a_n + a_{n-1}}{(n+2)(n+1)}$ $n \geq 1$

And we get

$$\begin{aligned} y(x) &= a_0\{1 + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{6} + \frac{(x-1)^4}{24} + \frac{(x-1)^5}{30} + \cdots\} \\ &+ a_1\{(x-1) + \frac{(x-1)^3}{6} + \frac{(x-1)^4}{12} + \frac{(x-1)^5}{120} + \cdots\} \end{aligned}$$

Hard to figure out a closed form formula for the a_n's.

Airy's Equation: Series Solution at $x_0 = 1$

Setting

- $y_1(x) = 1 + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{6} + \frac{(x-1)^4}{24} + \frac{(x-1)^5}{30} + \cdots$
- $y_2(x) = (x-1) + \frac{(x-1)^3}{6} + \frac{(x-1)^4}{12} + \frac{(x-1)^5}{120} + \cdots$

We'd like to be able to say what the radius of convergence of y_1 and y_2 is. However, since we cannot get a closed formula for the a_n's we cannot do this directly via the ratio test. We will see in the next part that we will be able to estimate the radius of convergence ...

Ordinary Point Revisited

Question

Do we really need to assume that P, Q and R are polynomials?

Definition

x_0 is said to be an ordinary point of the differential equation

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

if the functions $p(x) = \frac{Q(x)}{P(x)}$ and $q(x) = \frac{R(x)}{Q(x)}$ are analytic at x_0. Otherwise, we say that x_0 is a singular point.

Ordinary Point Revisited

Theorem

If x_0 is an ordinary point of the equation

$$P(x)y'' + Q(x)y' + R(x)y = 0,$$

then the general solution is of the form

$$y = \sum a_n(x - x_0)^n = a_0 y_1(x) + a_1 y_2(x),$$

where a_0 and a_1 are arbitrary and y_1 and y_2 are linearly independent series solutions centered at x_0. The radius of convergence of the Taylor series of the y_i's centered at x_0 is at least as large as the minimum of the radii of convergence of the Taylor series of p and q centered at x_0.

Ordinary Point Revisited

The solutions y_1 and y_2 in the previous theorem will be of the form:

$$y_1(x) = 1 + 0(x - x_0) + b_2(x - x_0)^2 + b_3(x - x_0)^3 + \cdots$$

and

$$y_2(x) = 0 + 1(x - x_0) + c_2(x - x_0)^2 + c_3(x - x_0)^3 + \cdots$$

- y_1 corresponds to the initial conditions

$$y(x_0) = 1, y'(x_0) = 0$$

- y_2 corresponds to the initial conditions

$$y(x_0) = 0, y'(x_0) = 1$$

A Useful fact

Proposition

Consider the rational function $h(x) = Q(x)/P(x)$, where P and Q are polynomials that do not have common factors. Then h is analytic at x_0 if and only if $P(x_0) \neq 0$. In the event that h is analytic at x_0, then the radius of convergence ρ of the Taylor series expansion of h at x_0 is given by

$$\rho = \min\{|x_0 - r_1|, \dots, |x_0 - r_k|\},$$

where $r_1, \dots, r_k$ are the roots of $P(x)$.

Remark

The polynomial $P(x)$ might have complex roots, in which case the distance computed above is the distance in the complex plane.

A Useful fact

Corollary

Consider the ODE $P(x)y'' + Q(x)y' + R(x)y = 0$*, where* P, Q *and* R *be polynomials (without common factors). If* x_0 *is a regular point of this ODE and* $r_1, \ldots, r_k$ *are the roots of* $P(x)$*, then the general solution of the ODE on an interval containing* x_0 *is of the form*

$$y(x) = a_0 y_1(x) + a_1 y_2(x),$$

where y_1 *and* y_2 *are analytic functions at* x_0 *and the radii of convergence of their respective Taylor series centered at* x_0 *are larger than* $\min\{|x_0 - r_1|, \ldots, |x_0 - r_k|\}$*.*

A Useful fact

Moral

The orevious corollary allows us to estimate the radius of convergence (and hence the interval of solution) of our ODE without explicitly calculating the a_n's and using the ration test. Indeed, recall Airy's equation

$$y'' - xy = 0.$$

This has an analytic solution $y(x) = \sum_{n=0}^{\infty} a_n(x-1)^n$ at $x_0 = 1$, but we previously noticed it was not possible to find a closed formula for the a_n's. However, since $p(x) = 0$ and $q(x) = -x = -1-(x-1)$ both have infinite radii of convergence at $x_0 = 1$, we see that the analytic solution $y(x)$ centered at $x_0 = 1$ will also have infinite radius of convergence.

Examples

Determine a lower bound for the radius of convergence of the series solution of

$$(x^2 - 2x - 3)y'' + xy' + 4y = 0$$

centered at

1. $x_0 = 4$;
2. $x_0 = 0$;
3. $x_0 = -4$.

Outline

Examples

Question

How do we analyze/solve 2nd order ODEs near singular points?

Euler's Equation

- Consider the homogeneous ODE

$$L[y] = x^2 y'' + \alpha x y' + \beta y = 0$$

- Has a singularity at $x = 0$
- Suppose the solution is of the form $y = x^r \equiv e^{r \ln(x)}$
- Then we get $L[x^r] = 0$ if and only if

$$F(r) = r(r-1) + \alpha r + \beta = 0$$

- But, $F(r) = (r - r_1)(r - r_2)$, where

$$r_1, r_2 = \frac{-(\alpha - 1) \pm \sqrt{(\alpha - 1)^2 - 4\beta}}{2}.$$

Euler's Equation: Real, Distinct Roots

- if $r_1 \neq r_2$ are real, then

$$W(x^{r_1}, x^{r_2}) = (r_2 - r_1)x^{r_1+r_2+1}.$$

  does not vanish for $x > 0$
- Hence $\{x^{r_1}, x^{r_2}\}$ is a fundamental set of solutions to the ODE on $x > 0$.

Example: Euler's Equation with Real, Distinct Roots

Solve the initial value problem

$$2x^2y'' + 3xy' - y = 0, \; y(1) = 1, y'(1) = 2, x > 0$$

- the General solution to ODE on $x > 0$ is

$$y(x) = c_1 x^{1/2} + c_2 x^{-1}.$$

- On $x > 0$ the IVP is satisfied by

$$y(x) = 2x^{1/2} - x^{-1}.$$

The Derivation: Equal Roots

- Recall that $x^r = e^{r\ln(x)}$, so $\frac{\partial}{\partial r} x^r = x^r \ln(x)$.
- Suppose $r_1 = r_2$, then $F(r) = (r - r_1)^2$.
- $\frac{\partial}{\partial r} L[x^r] = \frac{\partial}{\partial r}(x^r F(r))$
- $$\begin{aligned} L[x^r \ln(x)] &= L[\frac{\partial}{\partial r} x^r] \\ &= \frac{\partial}{\partial r} L[x^r] \\ &= x^r \ln(x)(r - r_1)^2 + 2(r - r_1)x^r \\ &= 0 \text{ (if } r = r_1) \end{aligned}$$

The Derivation: Equal Roots

- Hence, $x^{r_1}\ln(x)$ is a solution.
- $W(x^{r_1}, x^{r_1}\ln(x)) = x^{2r_1-1} > 0$ for $x > 0$
- Hence $\{x^{r_1}, x^{r_1}\ln(x)\}$ forms a fundamental set of solutions.
- The general solution in this case is

$$(c_1 + c_2\ln(x))x^{r_1}.$$

An Example: Euler's Equation with Equal Roots

Solve the following 2nd order IVP

$$x^2y'' + 5xy' + 4y = 0,\ \ y(1) = 2, y'(1) = 0, x > 0$$

- the General solution to ODE on $x > 0$ is

$$y(x) = x^{-2}(c_1 + c_2 \ln(x)).$$

- On $x > 0$ the IVP is satisfied by

$$y(x) = x^{-2}(2 + 2\ln(x)) = 2x^{-2}(1 + \ln(x)).$$

The Derivation: Complex Roots

- $r_1 = \lambda + i\mu$, $r_2 = \lambda - i\mu$
- So for $x > 0$

$$\begin{aligned} x^{r_1} &= e^{(\lambda+i\mu)\ln(x)} \\ &= e^{\lambda\ln(x)+i\mu\ln(x)} \\ &= e^{\lambda\ln(x)}e^{i\mu\ln(x)} \\ &= x^{\lambda}(\cos(\mu\ln(x)) + i\sin(\mu\ln(x))) \end{aligned}$$

and

$$x^{r_2} = x^{\lambda}(\cos(\mu\ln(x)) - i\sin(\mu\ln(x))).$$

The Derivation: Complex Roots

- $\{x^{r_1}, x^{r_2}\}$ forms a Fundamental set of solutions
- $\{x^\lambda \cos(\mu \ln(x)), x^\lambda \sin(\mu \ln(x))\}$ is a fundamental set of solutions consisting of real-valued functions.
- So the general solution is of the form

$$c_1 x^\lambda \cos(\mu \ln(x)) + c_2 x^\lambda \sin(\mu \ln(x)), \; x > 0.$$

Example: Euler's Equation with Complex Roots

Solve the IVP

$$x^2y'' + xy' + y = 0, \;\; y(1) = 0, y'(1) = 3, x > 0$$

- the General solution to ODE on $x > 0$ is

$$y(x) = c_1 \cos(\ln(x)) + c_2 \sin(\ln(x)).$$

- On $x > 0$ the IVP is satisfied by

$$y(x) = 3\sin(\ln(x)).$$

What about $x < 0$?

- In the three previous cases we restricted to the in interval $x > 0$.
- On the interval $x < 0$ we get

$$y(x) = \begin{cases} c_1|x|^{r_1} + c_2|x_2|^{r_2} \\ (c_1 + c_2\ln(|x|))|x|^{r_1} \\ c_1|x|^{\lambda}\cos(\mu\ln(|x|)) + c_2|x|^{\lambda}\sin(\mu\ln(|x|)) \end{cases}$$

Depending on the roots r_1, r_2 of $F(r) = r(r-1) + \alpha r + \beta$.

The Definition

Definition

Consider the second order ODE of the form

$$P(x)y'' + Q(x)y' + R(x)y = 0.$$

and let x_0 be a point where $P(x_0) = 0$. x_0 is said to be a regular singular point if

$$\lim_{x \to x_0} (x - x_0)\frac{Q(x)}{P(x)} \text{ and } \lim_{x \to x_0} (x - x_0)^2 \frac{R(x)}{P(x)}$$

are both finite. Otherwise, we say x_0 is an irregular singular point.

The Moral

- The singularity at $x_0 = 0$ of Euler's equation is regular.
- In general one can handle regular singular points in manner analogous to what we did for Euler's equation.
- We won't cover this, but it is useful in studying Bessel's equation.

Problems

1. Classify the singular points of the following ODE

$$2x(x-2)^2y'' + 3xy' + (x-2)y = 0.$$

2. Find the general solution to the following ODE that is valid on any interval not containing the singular point.

$$(x-1)^2y'' + 8(x-1)y' + 12y = 0$$