Math 23 Overivew

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Math 23 Differential Equations
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Outline

Outline

The Definition

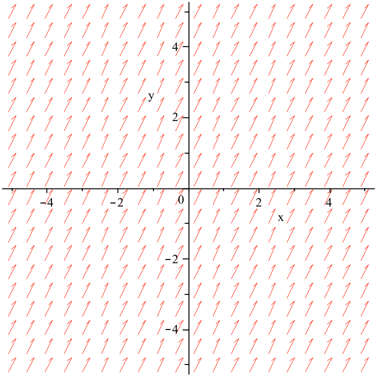
Definition

Let $D \subset \mathbb{R}^2$ be a planar region. A **vector field** on D is a mapping $\vec{\mathbf{F}}$ that assigns to each $(x, y) \in D$ a 2-dimensional vector

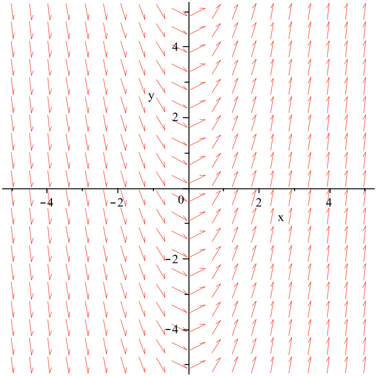
$$\vec{\mathbf{F}}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j},$$

where $P, Q : D \to \mathbb{R}$ are scalar functions. We will say that $\vec{\mathbf{F}}$ is **continuous** if P and Q are continuous.

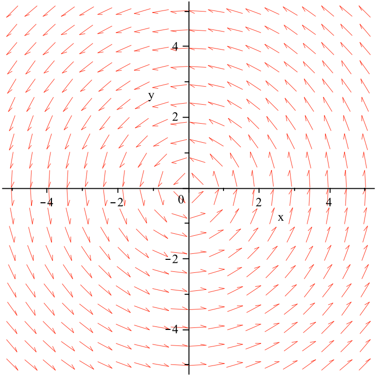
Example: $\vec{\mathbf{F}}(x, y) = \mathbf{i} + 2\mathbf{j}$



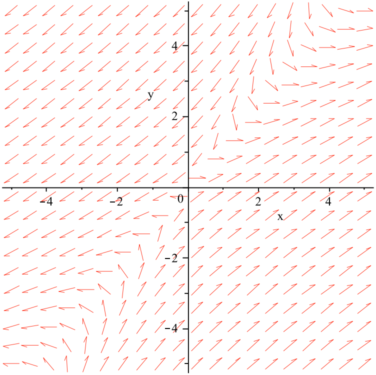
Example: $\vec{\mathbf{F}}(x, y) = \mathbf{i} + 2x\mathbf{j}$



Example: $\vec{\mathbf{F}}(x, y) = -y\mathbf{i} + x\mathbf{j}$



Example: $\vec{\mathbf{F}}(x, y) = (3x - 2y)\mathbf{i} + (2x - 2y)\mathbf{j}$



Vector Fields and Flow Lines

- The previous pictures seem to suggest motion
- If you dropped a particle into the Force field, the vectors should represent the velocity of the resulting motion.
- Lets formalize this ...

Vector Fields and Flow Lines

Definition

Let $\vec{\mathbf{F}}(x, y)$ be a vector field on $D \subset \mathbb{R}^2$. A curve $\gamma(t)$ in D is said to be a **flow line** (or **integral curve**) of $\vec{\mathbf{F}}$ if

$$\gamma'(t) = \vec{\mathbf{F}}(\gamma(t))$$

for all t.

Moral

The velocity at time t of the particle with motion described by $\gamma(t)$ is exactly $\vec{\mathbf{F}}(\gamma(t))$.

Vector Fields and Flow Lines

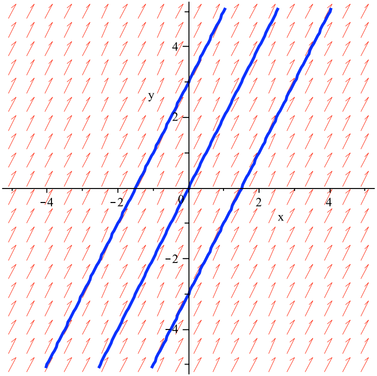
- Let $\vec{\mathbf{F}}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ be a vector field on $D \subset \mathbb{R}^2$.
- $\gamma(t) = (x(t), y(t))$ satisfies $\gamma'(t) = \vec{\mathbf{F}}(\gamma(t)$ if and only if

$$\begin{aligned} x'(t) &= P(x(t), y(t)) \\ y'(t) &= Q(x(t), y(t)) \end{aligned}$$

- the above is a system of Ordinary Differential Equations.

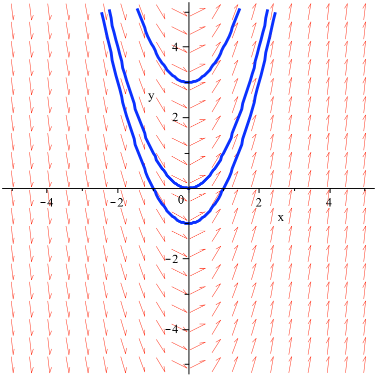
Example

The integral curves of $\vec{\mathbf{F}}(x, y) = \mathbf{i} + 2\mathbf{j}$ are of the form $\gamma(t) = (t + c_1, 2t + c_2)$



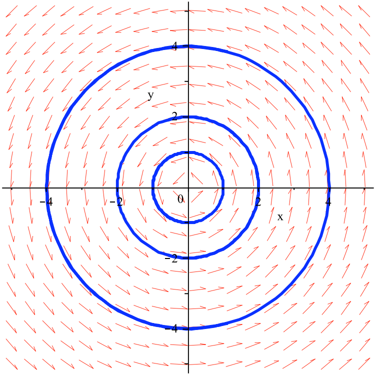
Example

The integral curves of $\vec{\mathbf{F}}(x, y) = \mathbf{i} + 2x\mathbf{j}$ are of the form $\gamma(t) = (t + c_1, t^2 + 2c_1 t + c_2)$



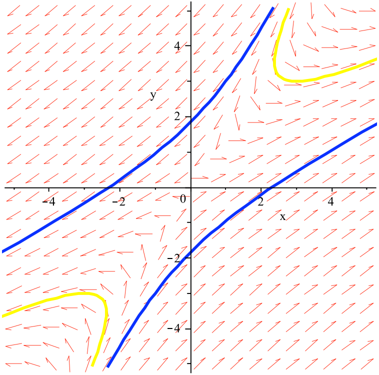
Example

The curves $\gamma(t) = (r\cos(t), r\sin(t))$ are integral curves of $\vec{\mathbf{F}}(x, y) = -y\mathbf{i} + x\mathbf{j}$.



Example

Some integral curves of $\vec{\mathbf{F}}(x, y) = (3x - 2y)\mathbf{i} + (2x - 2y)\mathbf{j}$.



Outline

The Problem

We want to describe the motion of a particle that is falling.

- let t denote the time (measured in seconds)
- Let v be the velocity of the particle (measured in m/s). It should be a function of time

$$v = v(t).$$

- We will assume that v is positive in the downward direction.
- Can we find an equation that governs v?

Newton's Second Law & Falling Bodies

- Newton's Second Law:

$$\mathbf{F} = ma,$$

  where m is the mass and a is the acceleration.
- $a = \frac{dv}{dt}$
- Hence,

$$\mathbf{F} = mv'(t).$$

- Now lets give an alternative description of **F**.

Newton's Second Law & Falling Bodies

- Clearly gravity acts on the particle by:

$$mg,$$

where m is the mass and g is the grav. const.

- Also, we know there is air resistance:

$$-\gamma v,$$

where γ is the drag coefficient for the body.

- Then

$$\mathbf{F} = mg - \gamma v.$$

Newton's Second Law & Falling Bodies

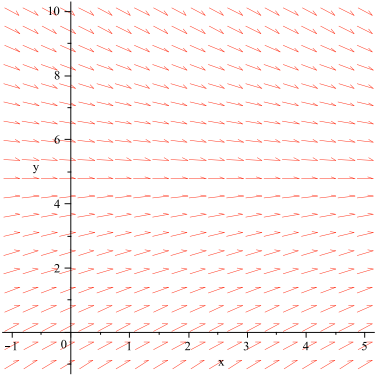
So Newton's Law forces us to conclude that we have the following relationship between t, v and v':

$$m\frac{dv}{dt} = mg - \gamma v.$$

This is an example of a differential equation.

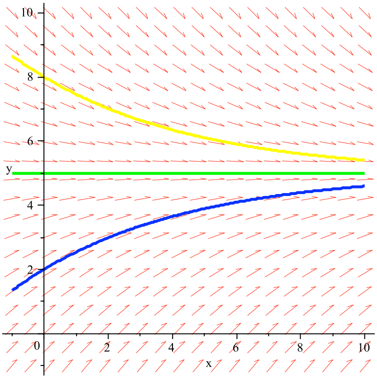
Can we see what this means?

Slope Fields & the Falling Body Problem



The slope field for $v' = 1 - 0.2v$

Slope Fields & the Falling Body Problem



Some solutions (or integral curves) of $v' = 1 - 0.2v$.

Outline

A Model

We want to analyze the population dynamics of a system consisting of two animals, one of which is the prey and the other the predator.
Here's a possible model:

- let t denote time in months
- Let $x(t)$ denote the number of prey at time t;
- Let $y(t)$ denote the number of predators at time t;

A Model

- In the absence of predators the population grows at a rate proportional to the current population:

$$\frac{dx}{dt} = ax,$$

for some $a > 0$.

- Similarly if $x(t) = 0$ for all t, then

$$\frac{dy}{dt} = -by,$$

for some $b > 0$. (Absence of food is a bad thing!)

A Model

- We also suppose that the number of interactions between predator and prey is proportional to the product of their populations:
$$x(t) \cdot y(t).$$
- Interactions are good for the predator and not so good for the prey:

$$\begin{aligned} \frac{dx}{dt} &= ax - \alpha xy \\ \frac{dy}{dt} &= by + \gamma xy \end{aligned}$$

A Model

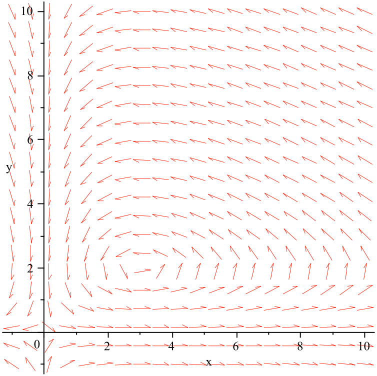
The pair of equations

$$\begin{aligned} \frac{dx}{dt} &= ax - \alpha xy \\ \frac{dy}{dt} &= by + \gamma xy \end{aligned}$$

is an example of a system of (non-linear) differential equations.

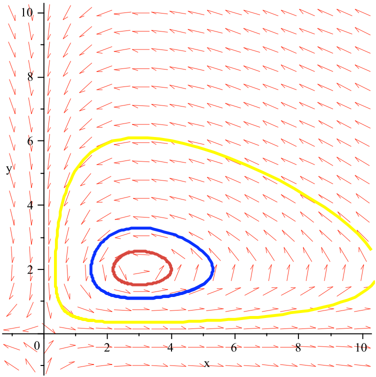
Can we visualize this?

In Pictures



The direction field for our predator-prey model

In Pictures



Some solutions (or integral curves) of our predator prey model

Outline

Basic Language

Definition

An n-th order ordinary differential equation is an expression of the form

$$F(t, y, y^{(1)}, y^{(2)}, \ldots, y^{(n)}) = 0,$$

where $y^{(j)}(t) = \frac{d^j y}{dt^j}(t)$ is the j-th derivative of y.

When considering ODEs we will usually think about those of the form

$$y^{(n)} = f(t, y, y^{(1)}, y^{(2)}, \ldots, y^{(n-1)})$$

Basic Language

Example

1. $ty + y^{(1)}y^{(3)} + y^{(2)}y^{(3)} + y^{(4)} = 0$
2. $cos(t)y + \ln(y') + y'' = 0$
3. $5y'' + 3y' - 7y = \sin(t)$

Basic Language

Definition

An n-th order ordinary differential equation is said to be **linear** if it is of the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y^1 + a_0 y = g(t).$$

Otherwise, we will say that it is **nonlinear**

We will see that there are many techniques for solving linear ODEs, but it is harder to solve nonlinear equations.

Basic Questions

1. How do you know when a solution exists?
2. If a solution exists:
   a) how do you find it?
   b) is it unique?
3. Can you infer qualitative information about the behavior of solutions from the equations (without solving explicitly)?