

Math 23: Differential Equations

Midterm Exam

May 2, 2018

Solutions

NAME: _____

SECTION (check one box) :

Section 1 (Kocyigit 11:30)

Section 2 (Kocyigit 12:50)

Instructions:

1. Wait for signal to begin.
2. Write your name in the space provided, and check one box to indicate which section of the course you belong to.
3. Please turn off cell phones or other electronic devices which may be disruptive.
4. Unless otherwise stated, you must justify your solutions to receive full credit. Work that is illegible may not be graded. Work that is scratched out will not be graded.
5. It is fine to leave your answer in a form such as $\ln(.02)$ or $\sqrt{239}$ or $(385)(13^3)$. However, if an expression can be easily simplified (such as $e^{\ln(.02)}$ or $\cos(\pi)$ or $(3 - 2)$), you should simplify it.
6. This exam is closed book. You may not use notes, or other external resource. You may use calculators. It is a violation of the honor code to give or receive help on this exam. However, you may ask the instructor for clarification on problems.
7. In True/False questions circle true, false or if you think the given information is not sufficient for a conclusion then write N/A and explain. Unless the question states "No explanation needed" you must explain your answer.
8. If you use a theorem you should check (and write clearly) whether the conditions of the theorem hold.

Honor statement: I have neither given nor received any help on this exam, and I attest that all of the answers are my own work.

Signature: _____

Problem	Score
1	8
2	9
3	9
4	10
5	8
6	8
7	8
8	10
9	10
10	10
11	10
12	10
13	10
Total	

{

60 pt

{ 20 pt

(Theory)
highest two scores

{ 20pt

(Laplace)
highest two scores

1. Consider the differential equations for $y(x)$

$$L[y] = 0 \quad (1)$$

$$L[y] = f(x) \quad (2)$$

where L is a linear differential operator and assume all coefficients of L and f are differentiable functions.
 Circle true or false for each statement. (No explanation needed.)

- [T / F] If y_p is a solution of (2) then Cy_p solves (2).
- [T / F] If y_p is a solution of (2) and y_h is a solution to (1) then $y_p + Cy_h$ solves (2).
- [T / F] If y_1 and y_2 are solutions of (2) then $y_1 - y_2$ solves (1).
- [T / F] If y_1 and y_2 are solutions of (2) then $y_1 + Cy_2$ solves (1) for any constant C .

2. Recall that we refer the constant solutions of differential equations as equilibrium solutions. Assume f, g are continuous functions. Circle true or false for each statement about the below equation for $y(x)$.

$$y' = y(x - y) \quad (1)$$

$$y' = f(x, y) \quad (2)$$

$$y' = g(y) \quad (3)$$

- [T / F] There exists an equilibrium solution of (1)

subs. $y=c$ into (1) and solve for c.

- [T / F] If $f(x, y)$ achieves 0 only at two points, say (x_0, y_0) and (x_1, y_1) , then there are no equilibrium solutions of (2)

*$y=c$ is a soln $\Rightarrow 0 = y' = f(x, c)$ for all x .
 f have infinitely many zeros.*

- [T / F] If $f(x, y)$ achieves 0 only at two distinct points, say (x_0, y_0) and (x_1, y_1) , then there are two equilibrium solutions of (2)

- [T / F] ~~If $g(y)$ has 4 zeros, then (3) has exactly 4 equilibrium solutions.~~

cancelled due to typo not corrected by everyone.

3. Given the below 3 equations

$$t^2 y' + 2ty + e^t = 0 \quad (1)$$

$$3y' + 4y + 5t = 0 \quad (2)$$

$$(1+t^2) y' + (1-e^t) (1+y^2) = 0 \quad (3)$$

Circle true or false. (No explanation needed, DE stands for differential equation)

[T / F] DE (1) is exact [T / F] DE (1) is linear [T / F] DE (1) is separable

[T / F] DE (2) is exact [T / F] DE (2) is linear [T / F] DE (2) is separable

[T / F] DE (3) is exact [T / F] DE (3) is linear [T / F] DE (3) is separable

4. Answer each part.

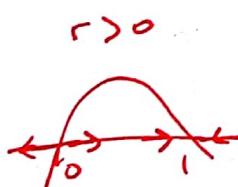
- (a) A person invests initial capital of $S_0 = \$12,750$ at an annual rate of return r compounded continuously. Write the differential equation model. It took 3 years for the original investment to double in value. Determine the sum $S(t)$ accumulated at any time t .

Answer: $\frac{dS}{dt} = rS, \quad S(t) = 12750 e^{\frac{\ln 2}{3} t}$

$$\begin{aligned} \frac{S'}{S} &= r \Rightarrow \ln S = rt \Rightarrow S(t) = S_0 e^{rt} \\ S(3) = 2S_0 &\Rightarrow 2S_0 = S_0 e^{r3} \Rightarrow r = \frac{\ln 2}{3} \end{aligned}$$

- (b) Suppose that a certain population obeys the logistic equation $\frac{dy}{dt} = ry[1-y]$. If $y_0 = 1/3$, find the time T at which the initial population has doubled. Find critical points and classify stability.

Answer: C.P.
0 unstable, 1 stable $T = \frac{\ln 4}{r}$



$$r > 0 \quad \frac{y'}{y(1-y)} = r \Rightarrow \frac{y'}{y} + \frac{y'}{1-y} = r \Rightarrow \ln|y| - \ln|1-y| = rt + C$$

$$\ln|\frac{y}{1-y}| = rt + C \Rightarrow \frac{y}{1-y} = e^c e^{rt}$$

$$\boxed{IC:} \quad \frac{1/3}{1-1/3} = e^c \Rightarrow e^c = 1/2$$

$$\text{doubling time} \quad \frac{2/3}{1-2/3} = (\frac{1}{2}) e^{rT} \Rightarrow 2 = \frac{1}{2} e^{rT} \Rightarrow T = \frac{\ln 4}{r}$$

5. Find general solution of

$$ty' + 2y = t^2.$$

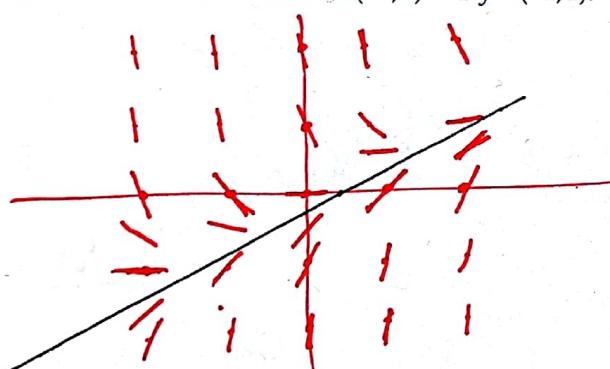
Answer: $y(t) = \frac{t^2}{4} + \frac{C}{t^2}, \quad t > 0$

$t > 0: \quad y' + \frac{2}{t}y = t \quad \text{integrating factor } \mu = e^{\int 2/t dt} = e^{2 \ln t} = t^2$
 $\Rightarrow t^2 y' + 2t y = t^3 \Rightarrow (t^2 y)' = t^3 \Rightarrow t^2 y = \frac{t^4}{4} + C$
 $\Rightarrow y(t) = \frac{t^2}{4} + \frac{C}{t^2}$

6. Consider the below equation for $y(x)$

$$y' = x - 2y$$

(a) Sketch the direction field in domain $x \in (-2, 2)$ and $y \in (-2, 2)$.



(b) Find the value of a, b such that $y = ax + b$ is a solution, and draw that solution on your sketch with a solid line.

Answer: $y = \frac{1}{2}x - \frac{1}{4}$

$$(ax+b)' = x - 2(ax+b)$$

$$a = (1-2a)x - 2b$$

$$\Rightarrow a = \frac{1}{2}, \quad b = -\frac{a}{2} = -\frac{1}{4}$$

7. (a) Find the general solution of

$$y'' + 3y' + 2y = 0$$

Answer: $y(t) = c_1 e^{-t} + c_2 e^{-2t}$

char. poly: $r^2 + 3r + 2 = 0 \quad r_1 = 1 \quad r_2 = 2$

e^{-t}, e^{-2t} are soln.

- (b) Use previous part to solve the IVP

$$y'' + 3y' + 2y = 5e^{-2t}, \quad y(0) = 2, \quad y'(0) = -8$$

Answer: $y(t) = e^{-t} + e^{-2t} - 5te^{-2t}$

undetermined coeff. $y_p = At^s e^{-2t}$

duplication $s=1$

$$y_p = At e^{-2t} \quad y_p' = Ae^{-2t} - 2At e^{-2t} \quad y_p'' = -4Ae^{-2t} + 4At e^{-2t}$$

substituting:

$$-4Ae^{-2t} + 4At e^{-2t} + 3Ae^{-2t} \cancel{+ 3.2At e^{-2t}} \cancel{+ 2At e^{-2t}} = 5e^{-2t}$$

$$\Rightarrow A = -5$$

gen soln $y(t) = c_1 e^{-t} + c_2 e^{-2t} - 5te^{-2t}$

L: $2 = y(0) = c_1 + c_2$

$$-8 = y'(0) = -c_1 - 2c_2 - 5e^0 + 10 \cdot 0 = -c_1 - 2c_2 - 5$$

$$\Rightarrow -1 = -c_2 \Rightarrow c_1 = 1, c_2 = 1$$

8. Circle true or false. If true then show, if false then explain or give a counter example, if the statement can't be concluded from the given information write N/A and explain.

F] If y_1 and y_2 are solutions of the equation

$$y' = \cos(t^2)y + t$$

and if the graphs of y_1 and y_2 intersect at some point then $W(y_1, y_2) = 0$ for all t .

Explain:

Suppose (t_0, y_0) is the intersection point.

IVP with initial condition $y(t_0) = y_0$

has a unique soln in \mathbb{R} since $\cos(t^2)$ on t are both continuous in \mathbb{R} . But y_1 and y_2 both solves this IVP $\Rightarrow y_1(t) = y_2(t)$ for all $t \in \mathbb{R}$

$$W(y_1, y_2) = W(y_1, y_1) = 0 \quad \text{for all } t.$$

F] There are infinitely many solutions of the equation

$$y'' + ty' + \sin(t)y = e^{t^2}$$

whose graphs intersect each other at origin.

Explain:

Consider IVP with initial condition $y(0)=0, y'(0)=A$

This has a unique soln since $t, \sin(t), e^{t^2}$ are all cont.

(Uniqueness and existence for 2nd order linear eqn. sec. 3.2)

Thus for each $A \in \mathbb{R}$ there is a different soln. which passes through origin ($y(0)=0$)

F] There exists a unique function that satisfies the IVP

$$y' = \frac{\sin(ty)}{t}, \quad y(1) = 0$$

on some interval $1-h < t < 1+h$.

Explain:

Direct statement of 1st order non-linear theorem

we check its conditions: $f(t, y) = \frac{\sin(ty)}{t}$

f and f_y are both continuous

say on rectangle $R = (+\frac{1}{2}, \frac{3}{2}) \times (-1, 1)$

which contains initial point $(1, 0)$

9. Let $p(t), q(t)$ be continuous functions on $t \in \mathbb{R}$

$$L[y] = y'' + py' + qy.$$

(a) Show that if u_1, u_2 are solutions to

$$L[y] = g(t)$$

then Wronskian $W(u_1, u_2)$ can achieve both positive and negative values (i.e. consequence of Abel's theorem does not hold for non-homogeneous problems). Hint: You can show by constructing a counter example, for example using DE $y'' = 2$. Explain:

$$L[y] = y'' = 2 \Rightarrow y(t) = t^2 + at + b \quad t \in \mathbb{R}$$

$$y_1(t) = t^2 \quad y_2(t) = t^2 + 1 \quad \text{are two soln}$$

$$W(y_1, y_2) = \begin{vmatrix} t^2 & t^2 + 1 \\ 2t & 2t \end{vmatrix} = 2t^3 - 2t^3 - 2t = 2t$$

which achieves negative and positive values in domain of soln.

(b) Let $y_1(t)$ and $y_2(t)$ be solutions of $L[y] = 0$ on $t \in \mathbb{R}$ such that

$$y_1(0) = 0 \quad \text{and} \quad y_1(1) = 0.$$

Show that if $\{y_1, y_2\}$ forms a fundamental set, then y_2 has to be 0 at some point $t \in (0, 1)$.

① $W(t) = W(y_1, y_2)(t)$ must be all positive or all negative in $t \in (0, 1)$

② $W(0) = -y'_1(0)y_2(0)$ and $W(1) = -y'_1(1)y_2(1)$ have same signs.

$y'_1(0)$ and $y'_1(1)$ can't be 0 by fact ①

③ Let's assume $y_1(t)$ doesn't vanish between $(0, 1)$

Then $y'_1(0)$ and $y'_1(1)$ have opposite signs

Then from ② $y_2(0)$ and $y_2(1)$ must also have opposite signs

\Rightarrow that implies $y_2(t)$ must achieve 0 in $t \in (0, 1)$

If $y_1(t)$ does vanish between 0 and 1 (i.e. assumption ③ doesn't hold)

then we can replace domain $(0, 1)$ with a

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smaller domain where ~~the~~ assumption ③ holds. (why?)

10. Given arbitrary differentiable functions $g(t), y_1(t)$ and $y_2(t)$, suppose that $W(y_1, y_2)$ is non-zero for all t .

- (a) Solve the below equations for $u_1(t)$ and $u_2(t)$. (Your can give your answer as an integral of known functions)

$$u'_1(t)y_1(t) + u'_2(t)y_2(t) = 0$$

$$u'_1(t)y'_1(t) + u'_2(t)y'_2(t) = g(t)$$

Answer: $u_1(t) = - \int \frac{+g(t)y_2(t)}{W(y_1, y_2)(t)} dt + C_1$ $u_2(t) = \int \frac{g(t)y_1(t)}{W(y_1, y_2)(t)} dt + C_2$

$$u'_1 = -\frac{y_2}{y_1} u_1' \Rightarrow -\frac{y_2}{y_1} y'_1 u_2 + y'_2 u_2' = g$$

$$\Rightarrow u'_2 = \frac{g}{y'_1 - \frac{y_2}{y_1} y'_1} = \frac{gy_1}{y_1 y'_2 - y'_1 y_2} \Rightarrow u'_1 = \frac{-gy_2}{y_1 y'_2 - y'_1 y_2}$$

Note: denominator is $W(y_1, y_2)$

- (b) Do you need the condition $W(y_1, y_2)$ to be non-zero in order to obtain the solution? Explain.

$W(y_1, y_2)$ is the determinant of the system above.
 If it is 0 we solve for u'_1 and u'_2 , and
 the integral might diverge.

11. Solve using Laplace transform

$$y'' + 4y' + 3y = \delta(t - 1), \quad y(0) = 0, \quad y'(0) = 0$$

Answer: $y(t) = \frac{1}{2} u_1(t) (e^{t-t} - e^{3(t-1)})$

$$s^2 Y + 4sY + 3Y = e^{-s} \Rightarrow Y(s) = \frac{e^{-s}}{s^2 + 4s + 3}$$

$$\frac{1}{s^2 + 4s + 3} = \frac{1}{(s+1)(s+3)} = \frac{1}{s+1} - \frac{1}{s+3}$$

$$\Rightarrow y(t) = \frac{1}{2} \mathcal{L}^{-1} \left\{ e^{-s} \left(\frac{1}{s+1} - \frac{1}{s+3} \right) \right\}$$

$\square : f \rightarrow u_1(t) f(t-1)$

$\boxed{\mathcal{L}} : F \rightarrow e^{-s} F(s)$

$$y(t) = \frac{1}{2} \square \mathcal{L}^{-1} \left\{ \frac{1}{s+1} - \frac{1}{s+3} \right\}$$

$$= \frac{1}{2} \square (e^{-t} - e^{-3t})$$

$$= \frac{1}{2} u_1(t) (e^{-(t-1)} - e^{-3(t-1)})$$

12. Solve $y'' + 4y = \begin{cases} 2, & 0 \leq t < \pi, \\ 0, & t \geq \pi; \end{cases}$ $y(0) = 2, \quad y'(0) = 0$

Answer: $y(t) = \frac{1}{2} + \frac{3}{2} \cos(2t) + u_{\pi}(t) [\cos(2t) - 1]$

$$\begin{matrix} 2 & \text{---} \\ 0 & \text{---} \\ & \pi \end{matrix} \quad 2 - 2u_{\pi}$$

$$\mathcal{L}: s^2 Y - s2 - 0 + 4Y = \frac{2}{s} - \frac{2e^{-\pi s}}{s}$$

$$Y(s) = \frac{2 - 2e^{-\pi s}}{s(s^2 + 4)} + \frac{2s}{s^2 + 4}$$

$$\frac{1}{s(s^2 + 4)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 4} \Rightarrow \begin{aligned} 1 &= (A+B)s^2 + Cs + 4A \\ C &= 0 \quad A = 1/4 \quad \Rightarrow \quad B = -1/4 \end{aligned}$$

$$y(t) = \mathcal{L}^{-1} \left\{ \frac{2}{4s} - \frac{2s}{4(s^2 + 4)} + \frac{2s}{s^2 + 4} \right\} + \mathcal{L}^{-1} \left\{ \frac{-2e^{-\pi s}}{4s} + \frac{2se^{-\pi s}}{4(s^2 + 4)} \right\}$$

$$y(t) = \mathcal{L}^{-1} \left\{ \frac{1}{2s} + \frac{6s}{4(s^2 + 4)} \right\} + \square \mathcal{L}^{-1} \left\{ \frac{-1}{2s} + \frac{s}{2(s^2 + 4)} \right\}$$

$$= \frac{1}{2} + \frac{3}{2} \cos(2t) + \square \left(-\frac{1}{2} + \frac{1}{2} \cos(2t) \right)$$

$$= \frac{1}{2} + \frac{3}{2} \cos(2t) + u_{\pi}(t) \left[-\frac{1}{2} + \frac{1}{2} \cos(2t - 2\pi) \right]$$

where
 $\square: f \rightarrow u_{\pi}f(t-\pi)$

13. Solve the following IVP. You can leave your answer as an integral of known functions.

$$y'' - 2y' + y = e^{t^2}, \quad y(0) = 0, y'(0) = 0$$

1st way

Answer: $y(t) = \int_0^t (t-z) e^{t-z} e^{z^2} dz$

$$\int : s^2 Y - 2sY + Y = G(s)$$

formally

$$Y(s) = \frac{G(s)}{s^2 - 2s + 1} = H(s)G(s)$$

$$h(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2 - 2s + 1}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t^0 t = t e^t$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\{H \cdot G\} = (f * g)(t) = \int_0^t (t-z) e^{t-z} e^{z^2} dz$$

Note 1st step is formally since $\mathcal{L}\{e^{t^2}\}$ doesn't exist. But we can instead cut-off using $g(t) = x_T \cdot e^{t^2}$ where $x_T = 1 - u_T$, then solve and take $T \rightarrow \infty$.

$\square : F \rightarrow F(st)$

$\square : f \rightarrow e^{-t} f$

13. Solve the following IVP. You can leave your answer as an integral of known functions.

$$y'' - 2y' + y = e^{t^2}, \quad y(0) = 0, y'(0) = 0$$

2nd way

Answer: $\int_0^t (t-z) e^{t-z} e^{z^2} dz$

$$W(e^t, t e^t) = e^{2t}$$

Homogeneous eqn. $r^2 - 2r + 1 = (r-1)^2 = 0 \Rightarrow y_1 = e^t, y_2 = te^t$

Using variation of parameters (u_1, u_2 are as in question #10)

$$y(t) = c_1 e^t + c_2 te^t + y_p(t) \quad \text{where} \quad y_p(t) = y_1 u_1 + y_2 u_2$$

$$= -e^t \int_0^t z e^z e^{z^2} dz + te^t \int_0^t e^z e^{z^2} dz$$

$$y_p(t) = \int_0^t (t-z) e^{t-z} e^{z^2} dz$$

IC $\left. \begin{array}{l} 0 = y(0) = c_1 + 0 + 0 \\ 0 = y'(0) = c_1 + c_2 + 0 \end{array} \right\} \quad \begin{array}{l} c_1 = 0 \\ c_2 = 0 \end{array}$

Thus $y(t) = y_p(t) = \int_0^t (t-z) e^{t-z} e^{z^2} dz$