Math 23, Spring 2018

Dartmouth College

Week 3

Exact equations

Example

$$cos(x)y^2 + sin(x)2yy' = 0, \qquad \pi/2 < x < \pi$$

$Link : Notes(B3.1)$

Definition

An equation

$$M(x,y) + N(x,y)y' = 0$$

is called exact, if there exists a $\psi(x,y)$ such that

$$M = \frac{\partial \psi}{\partial x} \quad N = \frac{\partial \psi}{\partial y}.$$

Exact equations

If it is exact then, then letting $y = \phi(x)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}x}\psi(x, \phi(x)) = 0$$

or equivalently

$$\psi(x, \phi(x)) = C$$

Then $y = \phi(x)$ is a solution to the above ODE.

Exercise: Verify the converse using chain rule.

Exact equations, Exercise 1

Two questions

- How to use exactness to solve DE?

  Using exactness obtain an equation $\psi(x, y) = C$.
  which is not a differential eqn!
- How to check if a given DE is exact or not?

Exercise 1

$$(2x + 3) + (2y - 2)y' = 0$$

is exact. Find M, N, and Ψ.

Main theorem

Theorem 2.6.1

Let the functions $M, N, M_y := \frac{\partial M}{\partial y}, N_x := \frac{\partial N}{\partial x}$ be continuous on the rectangular region $R = (\alpha, \beta) \times (\gamma, \delta)$. Then

$$M(x, y) + N(x, y)y' = 0$$

is exact on R if and only if

$$M_y(x, y) = N_x(x, y) \tag{1}$$

at all points (x, y) in R. In other words, (1) holds is and only if there is a function ψ such that

$$\psi_x(x, y) = M(x, y) \qquad \psi_y(x, y) = N(x, y). \tag{2}$$

Exercise 1, cont'd

Solve the IVP

$$(2x + 3) + (2y - 2)y' = 0, \quad y(1) = 0$$

$\boxed{Link : Notes(B3.1)}$

Integration factors

Note

$$\mu \text{ only depends on } x \iff \mu \text{ is a function of only } x \iff \mu_x = 0$$

- If DE is not exact can we make it exact. For example

$$cos(x)y + sin(x)2y' = x/y, \qquad \pi/2 < x < \pi$$

is not exact but

$$cos(x)y^2 + sin(x)2yy' = x, \qquad \pi/2 < x < \pi$$

is exact.

- Suppose

$$M_y \neq M_x$$

but

$$(\mu M)_y = (\mu N)_x$$

for some $\mu = \mu(x, y)$

Exact equations

Example

$$cos(x)y + sin(x)2y' = 0, \qquad \pi/2 < x < \pi$$

$Link : Notes(B3.1)$

Integration factors

We consider 3 special cases

- If $\dfrac{M_y - N_x}{N}$ only depends on x, then
$$\frac{\mu'(x)}{\mu(x)} = \frac{M_y - N_x}{N}$$
- If $\dfrac{N_x - M_y}{M}$ only depends on y, then
$$\frac{\mu'(y)}{\mu(y)} = \frac{N_x - M_y}{M}$$
- Exercise 24 covers $\mu = \mu(xy)$

Second order linear equations

- **Second order** equation

$$\Rightarrow y'' = f(t, y, y')$$

- Second order **linear equation**

$$\Rightarrow y'' + p(t)y' + q(t)y = g(t)$$

Homogeneous equations and initial conditions

Definition

A second order linear equation

$$y'' + p(t)y' + q(t)y = g(t)\,.$$

is called **homogeneous** if $g(t) = 0$.
Otherwise it is called **nonhomogeneous**.

Note For second-order IVP' **two** initial conditions needed. As an example consider $y' = 0$. Consider its general solution, how many IC do you need?

Constant coefficient DE's

Observation: Substituting e^{rt} into

$$ay'' + by' + cy = 0 \qquad (DE)$$

gives

$$ar^2 + br + c = 0 \qquad \text{(characteristic polynomial)}$$

Three verifications :

- Verify above as an exercise
- Verify : If r_1, r_2 are distinct solutions of quadratic equation then $y_1 = e^{r_1 t}, y_2 = e^{r_2 t}$ are solutions of (DE).
- Verify that $\alpha y_1 + \beta y_2$ is a solution of (DE) for any α, β

NOTE: More generally, when can we add solutions to obtain new solutions ? See below (also recall HW problem 2.4 Q25)

Examples

Find general solution of

$$y'' - y = 0$$

How many solutions does below IVP have?

$$y'' - y = 0, \qquad y(0) = 2$$

Solve IVP

$$y'' - y = 0, \qquad y(0) = 2, y'(0) = 0$$

$\boxed{Link : Notes(B3.2)}$

Constant coefficient DE's

We can solve DE's of the form

$$ay'' + by' + cy = 0$$

when characteristic polynomial have two distinct roots $r_1 \neq r_2$.

Solve IVP

$$2y'' - 3y' + y = 0, \qquad y(0) = 1, y'(0) = 1$$

Constant coefficient DE's

We can solve DE's of the form

$$ay'' + by' + cy = 0$$

when characteristic polynomial have two distinct roots $r_1 \neq r_2$.

Exercises

- We found two distinct solutions $\to$ two equations $\to$ determined two constants
- What happens char. poly. has repeated roots?
- Analyze the long term behavior of the soln of above DE if its char. poly. has two distinct roots r_1, r_2 :
  - Does the long term behavior depend on the initial conditions?
  - Possible outcomes:Grows exponentially, stays constant or decays to 0.

Note: We'll handle the other cases later (repeated roots or no-real roots).

Examples

General solution of

$$y'' - y = 0$$

is

$$y(t) = \alpha e^{t} + \beta e^{-t}$$

What does the solution look "like" as $t \to \infty$?
If $y(0) = a, y(0) = b$ how does the limit of the solution as $t \to \infty$ depends on a, b?

Existence and uniqueness

Define differential operator $L = D^2 + pD + q$ by

$$L[y] = (D^2 + pD + q)y = y'' + py' + qy$$

where

$$D : \text{derivative operator}$$

and

$$p, q : \text{continuous functions of t}$$

Existence and uniqueness

Theorem 3.2.1

Consider the IVP

$$y'' + p(t)y' + q(t)y = g(t), \qquad y(t_0) = y_0, \qquad y'(t_0) = y_0'$$

where p, q, g are continuous on (α, β) and $t_0 \in (\alpha, \beta)$. Then, there is **exactly one solution** $y(t)$ of this problem, and the solution is defined **on the whole interval** (α, β).

- Here y is twice differentiable (y, y', y'' are all continuous).
- Apply to constant coefficient homogeneous case: Unique solution on $\mathbb{R}$

Existence and uniqueness

Theorem 3.2.1

Consider the IVP

$$y'' + p(t)y' + q(t)y = g(t), \qquad y(t_0) = y_0, \qquad y'(t_0) = y_0'$$

where p, q, g are continuous on (α, β) and $t_0 \in (\alpha, \beta)$. Then, there is exactly one solution $y(t)$ of this problem, and the solution is defined on the interval (α, β).

Exercise. Find longest interval in which the solution of below IVP's are certain to exist.

$$(t-1)y'' + cos(t)y' + e^t y = t, \qquad y(0) = y_0, \qquad y'(0) = y_0'$$

$$cos(t)y'' + ty' + e^t y = sin(t), \qquad y(0) = y_0, \qquad y'(0) = y_0'$$

Superposition

Theorem 3.2.2

If y_1 and y_2 are solutions of the differential equation

$$L[y] = P(t)y'' + Q(t)y' + R(t)y = 0$$

then the linear combination $c_1y_1 + c_2y_2$ is also a solution for any constants c_1, c_2.

- Crucial property. Observe how linearity of the DE plays a role.

Exercise. In order to see how homogeneity plays a role: Which DE do each function $z(t)$ solve?

- $z(t)$ is sum of a solution of $L[y] = 0$ and a solution of $L[y] = g(t)$.
- $z(t)$ is sum of a solution of $L[y] = f(t)$ and a solution of $L[y] = g(t)$.

Wronkskian

Definition

Let y_1 and y_2 be two solutions of the homogeneous ODE

$$y'' + p(t)y' + q(t)y = 0.$$

The **Wronkskian** of these solutions is a function in t

$$W(y_1, y_2)(t) := \det \begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix} = y_1(t)y_2'(t) - y_2(t)y_1'(t)$$

Exercise 20 (practice)

$W(f, g)(t) = t\cos t - \sin t$;
$u = f + 3g$ and $v = f - g$.

Find $W(u, v)(t)$.

Theorem 3.2.3

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Let $y_1(t)$ and $y_2(t)$ be two solutions of the homogeneous ODE $y'' + p(t)y' + q(t)y = 0$. Then

$$W(y_1, y_2)(t_0) \neq 0$$

if and only if

$$y(t) = C_1 y_1(t) + C_2 y_2(t) \qquad (*)$$

is a general solution.
Thus any IVP can be solved by choosing constants C_1 and C_2.

- Recall. General solution is a family of solutions that includes any solution of the given DE.
- "a" general solution?

Theorem 3.2.4

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Let $y_1(t)$ and $y_2(t)$ be two solutions of the homogeneous ODE $y'' + p(t)y' + q(t)y = 0$. Then

$$W(y_1, y_2)(t_0) \neq 0 \text{ at some point } t_0$$

if and only if

$$y(t) = C_1 y_1(t) + C_2 y_2(t) \qquad (*)$$

is a general solution.

- Compare the last two theorems.

Abel's Theorem

Abel's Theorem (3.2.7)

If y_1, y_2 are solutions on the interval (α, β) of

$$y'' + p(t)y' + q(t)y = 0,$$

and $p(t), q(t)$ are continuous on interval I.
Then,

$$W(y_1, y_2)(t) = Ce^{-\int p(t)\,\mathrm{d}t} \quad \text{for } t \in I$$

In particular, $W(y_1, y_2)$ is either 0 or never achieves 0.

- This theorems implies that Theorems 3.2.3 and 3.2.4 are essentially same. Why?

Link : *Notes*(*B*3.2)

Summary of 4 results

Problem 7 Midterm 1, Winter 2014

a) Find two constants n such that $y = t^n$ is a solution to the differential equation

$$t^2y'' + 3ty' - 3y = 0$$

b) Write down a general solution for $t < 0$.

- What is the domain of the solutions you find?

Fundamental Solution set

Definition

Assume that $p(t)$ and $q(t)$ are **continuous** in (α, β). Further, assume that $y_1(t)$ and $y_2(t)$ are two solutions in (α, β) for

$$y'' + p(t)y' + q(t)y = 0.$$

If $W(y_1, y_2)(t_0) \neq 0$ for **some** $t_0 \in (\alpha, \beta)$ then $\{y_1(t), y_2(t)\}$ is called the **fundamental solution set** of the ODE on the interval (α, β).

Exercise: Prove Theorems 3.2.5 and 3.2.6 using above 4 results.

Homogeneous equations with constant coefficients

$Link$: $Notes(B3.3)$

Complex numbers

Euler's identity

Complex and repeated roots of char. poly

Summary

If the characteristic equation associated to $ay'' + by' + cy = 0$ has

1. **two different real roots** $r_1, r_2 \in \mathbb{R} \Rightarrow y = C_1 e^{r_1 t} + C_2 e^{r_2 t},\ C_1, C_2 \in \mathbb{R}$
2. **double root** $r \in \mathbb{R} \Rightarrow y = C_1 e^{rt} + C_2 t e^{rt},\ C_1, C_2 \in \mathbb{R}$
3. **two complex roots** $r = \alpha \pm i\beta \Rightarrow y = C_1 e^{\alpha t} \cos(\beta t) + C_2 e^{\alpha t} \sin(\beta t)\ C_1, C_2 \in \mathbb{R}$

Exercise 18

Solve the IVP: $\begin{cases} y'' + 4y' + 5y = 0 \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$

1. $r^2 + 4r + 5 = 0 \rightsquigarrow r = \frac{-4 \pm \sqrt{16-20}}{2} = -2 \pm i$
2. $y_1(t) = e^{-2t}\cos(t)$
   $y_2(t) = e^{-2t}\sin(t)$
   $y(t) = c_1 e^{-2t}\cos(t) + c_2 e^{-2t}\sin(t)$
3. Solve for c_1 and c_2
4. $y(t) = e^{-2t}\cos(t) + 2e^{-2t}\sin(t)$
5. What happens as $t \to +\infty$

Reduction of order method

Let y_1 be a solution to

$$y'' + p(t)y' + q(t)y = 0$$

Reduction of order : Want to check whether $v(t)y_1(t)$ gives a "new" solution: Substitute $v(t)y_1(t)$ in the equation and solve for v such that vy_1 is a different solution than y_1.

$$v''y_1 + v'(2y_1' + py_1) = 0$$

$$w'y_1 + (2y_1' + py_1)w = 0, \quad w = v'$$

and we know how to solve for w, and $v = \int w$

Exercise

Exercise 3.4.24

Solve $ty'' + 3ty' - 3y = 0$, given $y_1 = t$

$\boxed{Link : Notes(B3.3)}$

Exercise 3.4.24

Solve $t^2y'' + 2ty' - 2y = 0$, given $y_1 = t$

Exercise 16

a) Find a solution to the initial value problem as a function of b

$$y'' - y' + \frac{1}{4}y = 0, \quad y(0) = 2, \quad y'(0) = b$$

b) Determine a critical value of b that separates solutions that grow positively from those that eventually grow negatively.

$$y(t) = e^{t/2}(bt - t + 2)$$
$$y'(t) = \frac{1}{2}e^{t/2}((b-1)t + 2b)$$