Math 23, Spring 2018

Dartmouth College

Week 2

Gluing Solutions

Example 2.4.3

Solve the IVP : $y' = y^{1/3}, \quad y(x_0) = 0$

Question

Are there more solutions?

For $x_0 > 0$ put

$$y^{\pm}_{x_0}(x) = \begin{cases} 0 & x \leq x_0 \\ \pm\sqrt{\left(\frac{2}{3}(x - x_0)\right)^3} & x > x_0 \end{cases}$$

- Conversely, suppose above is a family of solutions for some IVP. Can this IVP satisfy conditions of the Theorem 2.4.2?

Discontinuous Coefficients

Following problem involves a coefficient with a jump discontinuity. In such cases:

- Solve equation on each interval where coeffients are continuous
- Impose conditions so that y is a continuous function.

NOTE: Derivative of a continuous (and piecewise differentiable) function is a "function". However derivative of a discontinuous function is more complicated (might not be a "function") and discontinuous solutions are out of our context. Therefore solutions we want (unless otherwise specified) will be **continuous**.

Discontinuous Coefficients

Solve

$$y' + 2y = g(t) = \begin{cases} 1, & 0 \le t \le 1 \\ 0, & t > 1 \end{cases}, \quad y(0) = 0$$

1. Solve $y_1' + 2y_1 = 1, \quad y_1(0) = 0, \quad t < 1$
2. Solve $y_2' + 2y_2 = 0, \quad y_2(1) = y_1(1), \quad t > 1$
3. Then $y(t) = \begin{cases} y_1(t) & 0 \le t \le 1 \\ y_2(t) & t > 1 \end{cases}$ is a solution for the IVP, even though $y'(t)$ is not continuous.

Link : *Notes*(*B*2.1) `Two Modelling examples`

Modeling with differential equations

Three important steps in modeling:

- Construction of the model
- Analysis of the model
- Comparison with experiment or observation

Some tests of your model

- Check units, (same units: summands, left and right hand side of equations, exponents are unitless)
- Check for non-physical situations (for example negative or infinite quantity where you don't expect)
- Do simplifying what-if tests (what happens if rate-of-change r is zero? Does this comply with the pyhsics?)
- Long term behaviour of your model

Modeling with differential equations

Example 2.3.1

At time $t = 0$ a tank contains Q_0 lb of salt dissolved in 100 gal of water. Assume that water containing 0.25lb of salt/gal is entering the tank at a rate of r gal/min and that well-stirred mixture is draining from the tank at the same rate.

1. Setup IVP that describes this flow process.
2. Find the amount of salt $Q(t)$ in the tank at any time
3. Find limiting amount Q_L that is present after a very long time
4. Set $r = 3$, $Q_0 = 2Q_L$ and find the time T after which the salt level is within 2% of Q_L.
5. Find flow rate r that is required if T is 45min

- Setting your variables and unknown function is crucial step. Do we need concentration at time t in the above problem? See second problem below.
- Can you answer 3 without solving? (to use it as a verification of our model)

Modeling with differential equations

What will change if rate of change r is not constant but it is a function of time?

$Link : Notes(B2.2)$
Compound Interest

Autonomous equations and population dynamics

Autonomous equations

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(y)$$

Basic intuition :

- value of f determines rate of change (for ex positive / negative / zero)
- f depends only on population y (what will direction field look like)

Link : *Notes*(*B*2.3)

Critical points, equilibrium soln and stability

Logistic growth models

Example

$$y' = f(t, y)$$

Which of the following are true about the above DE

1. If $y'(1) = 0$ then $y(t) = 0$ is a constant solution (ie. equilibrium solution).
2. If $y'(1) = 0$ then $y(t) = 1$ is a constant solution
3. $y(t) = 1$ is a solution then $f(t, 1) = 0$ for all t.
4. First statement is true if f depends only on y
   (ie. f is independent of t or in other words DE is autonomous)
5. Second statement is true if f depends only on y (ie. DE is autonomous)

Answers: F, F, T, F, T

Example

Suppose $0, 1$ are the only two critical points of

$$y' = f(y).$$

Which of the following are true.

1. If $f'(0) < 0$ then soln $y(t) = 0$ is asymptotically stable
2. If $f'(1) > 0$ then soln $y(t) = 1$ is asymptotically unstable
3. If $f'(0) < 0$ then soln $y(t) = 1$ is asymptotically unstable
4. Third statement is true if $f(y)$ is a quadratic polynomial

Answers: T, T, F, T