# Math 23, Spring 2018

Dartmouth College

Week 1

# Math 23

- Tutorials: Su, Tu, Th : 7 pm - 9 pm
  Kemeny 105
- Homework: Due Wednesday's 3:30 pm
  - first one due next week
  - on the boxes on the 1st floor
- Office hours:
  - TBA

# Differential Equations

- Differential Equation : Equations containing derivatives of a function
- Example

$$\frac{dy}{dt} = y + 10 \quad (DE)$$

- What does it mean to solve a differential equation, such as (DE) above?

  Find a function y(t) which satisfies (DE)
- More general example :

$$\frac{dy}{dt} = f(t, y)$$

- A few initial questions we will consider :
  - Is there a solution?
  - How "many" solutions are there?
  - Is there a solution that is independent of time (i.e. does not change with time)?

- Definition. Equilibrium solution: a solution that is independent of time

$$\frac{dy}{dt} = f(t, y)$$

- $f$ is known so we know $\frac{dy}{dt}$ at any given point $(t, y)$.
- We can plot it on t-y plane: it is called direction field or slope field.

# Direction Field

Consider

$$\frac{dv}{dt} = 10 - \frac{v}{5}$$

Link: DirectionField.nb

$\boxed{Link : Notes(B1.1)}$

# Some examples

*Link : examplesDE.pdf*

# Solutions of some DE's

We consider

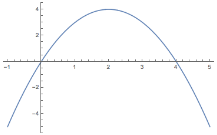
$$\frac{dp}{dt} = \alpha p - K$$

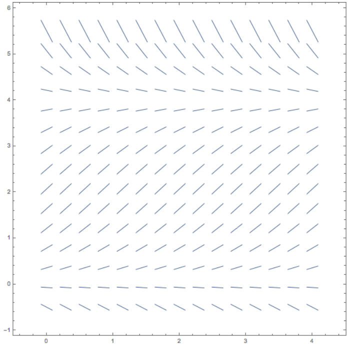
$\boxed{Link : Notes(B1.2)}$

- Solution of above equation
- Definition: Equilibrium solution
- Definition: General solution

Draw the direction field of $y' = y(4 - y)$.

$$\frac{dy}{dt} \text{ vs } y$$




$$y = \frac{4e^{4t}}{D + e^{4t}} \qquad \textbf{(general solution)}$$

# Classification of Differential Equations

Some vocabulary so we can describe different situations that arise

- **Ordinary** Differential Equations vs **Partial** Differential Equations
- What is a system of Differential Equations?
- **Order** of a Differential Equation
- **Linear** vs Nonlinear
- Formal definition of a solution

# Ordinary vs Partial

**Definition**

An **ordinary differential equation** (ODE) is a differential equation containing one or more functions of one independent variable and its derivatives.
A **partial differential equation** (PDE) is a differential equation that contains unknown multivariable functions and their partial derivatives.

Examples

- $\dfrac{dv(t)}{dt} = 10 - \dfrac{v(t)}{5}$ is an ODE, $v(t)$ only depends on $t$.
- $a^2 \dfrac{\partial^2 u(x,t)}{\partial x^2} = \dfrac{\partial^2 u(x,t)}{\partial t^2}$ is a PDE, $u(x,t)$ depends on $x$ and $t$.
- Below equation is also an ODE.

$$\frac{\partial^2 u(x,t)}{\partial x^2} + \frac{\partial u(x,t)}{\partial x} = 1$$

# Systems of DEs

We have only considered a single differential equation, sometimes there are models that involves several functions, all connected by DEs, known as **systems of DEs**.
For example,

$$\frac{dx}{dt} = ax - \alpha xy$$
$$\frac{dy}{dt} = -cy + \gamma xy$$

- $t$ is the independent variable.
- $x, y$ are **unknown** functions of independent variable $t$,
- $a, \alpha, c, \gamma$ are constants

# Order of DEs

**Definition**

The **order** of a differential equation is the order of the highest derivative that appears in the equation.

- First order ODE: $\dfrac{dy}{dt} + \dfrac{2}{t}y = 4t$
- Third order ODE: $y''' + 2e^t y'' + y'y = t^4$
- More generally, an $n^{\text{th}}$ order ODE

$$F(t, y, y', \ldots, y^{(n)}) = 0$$

- Example. Can you transform below equation to a second order DE?

$$y'''' - y'' = t$$

# Linear vs nonlinear

**Definition**

An ordinary differential equation

$$F(t, y, y', \ldots, y^{(n)}) = 0$$

is **linear** if $F$ is a linear function of the variables $y, y', \ldots, y^{(n)}$.

- If a DE is not linear then it is called **nonlinear**.
- In general linear ODE of order $n$ is

$$a_n(t)y^{(n)} + \cdots + a_1(t)y' + a_0(t)y = g(t)\,.$$

# Linear vs nonlinear

Classify each DE

- $y' = y^2$
- $y' = \sin y$
- $e^t y'' + t^3 y + e^{sin(t^2)+1} = 0$
- $(t^3 - 1)(y' + y) + 1 = t$
- first two nonlinear last two are linear.

# Solutions

**Definition**

A **solution** to an ODE $F(t, y, \ldots, y^{(n)}) = 0$ on an interval $(a, b)$ is a function $y(t)$ that satisfies this equation (i.e., that makes it true) for all $t$ in $(a, b)$.

- No solution: There is no real valued function $y(t)$ that satisfies the equation on the given interval.
- Q: Do all DEs have solutions?
  A: No, not necessarily. $(y')^2 + 1 = 0$;
- Q: Can a DE have more than one solution?
  A: We have seen that this is often the case. We'll see later in the course that General solution will tell us "how many" solutions a given DE can have.

# Solving linear ODEs

## Example 1

Solve

$$(1+t^4)y' + 4t^3y = t$$

- Hint: left hand side looks like it is derived using product rule
- Using above can we solve :

$$y' + \frac{4t^3}{1+t^4}y = g(t)$$

- How about

$$y' + p(t)y = g(t)$$

$\boxed{Link : Notes(B1.3)}$

$$y' + p(t)y = g(t)$$

- We multiply with an "appropriate" function.
- Note: $g$ can be a complicated function of t. (doesnot affect the method)

**Exercise 13**

Solve the Initial Value Problem (IVP)

$$\frac{ty'}{4} + y = sin(t^4), \quad y(\pi^{1/4}) = 0, \quad t > 0$$

# Some Questions

- Suppose you can't calculate

$$\int \mu(t)g(t)\,\mathrm{d}t$$

or

$$\mu = \int p(t)\,\mathrm{d}t$$

is this method still useful?

# Summary

- We learned some vocabulary, ordinary/partial, linear/nonlinear, order, etc..
- We learned how to solve $y' + p(t)y = g(t)$
- The general solutions looks like

$$y(t) = \frac{1}{\mu(t)}\left[\int \mu(t)g(t)\,\mathrm{d}t + C\right]$$

with $\mu(t) = e^{\int p(t)\,\mathrm{d}t}$

# Separable Equations

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$$

- If linear then use method of integrating factors
- If non-linear no single method.

We now see a method that can solve a special type of equations (linear or non-linear) : Separable

# Separable Equations

Consider

$$M(x,y) + N(x,y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \tag{1}$$

where functions $M, N$ can be non-linear.

If DE can be rewritten as

$$M(x) + N(y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0\,.$$

we say it is **separable**.

# Separable equations

If separable: we can solve it simply integrating w.r.t. x

$$\int N(y)\,\mathrm{d}y = -\int M(x)\,\mathrm{d}x.$$

Then solve for $y$.

$\boxed{Link : Notes(B1.4)}$

# Separable equations, Examples

Determine order, linearity and separability of the following IVPs and then solve each.

**Example**

$$x^2y^3 - y' = 0, \quad y(0) = 1$$

**Example**

This IVP has multiple solutions . Find a solution of the IVP

$$x^2y' = 1 + x^2 + y^2 + x^2y^2, \quad y(1) = 0$$

$\boxed{Link : Notes(B1.4)}$

# Existence and Uniqueness – linear ODEs

**Theorem 2.4.1**

Suppose the functions $p, g$ are continuous on the interval $(\alpha, \beta)$ containing the point $t_0$. Then there exists a unique function $y(t)$ that satisfies the DE

$$y' + p(t)y = g(t)$$

for each $t$ in $(\alpha, \beta)$ and also satisfies the initial condition $y(t_0) = y_0$, where $y_0$ is an arbitrary initial value.

- Can two solutions intersect at a point $t \in (\alpha, \beta)$ ?

# Existence and Uniqueness – non-linear ODEs

**Theorem 2.4.2**

Suppose the functions $f(t, y)$ and $\dfrac{\partial f}{\partial y}(t, y)$ are continuous on some rectangle $(\alpha, \beta) \times (\gamma, \delta)$ containing the point $(t_0, y_0)$.
Then, in some interval $(t_0 - h, t_0 + h)$ contained in $(\alpha, \beta)$, there is a unique solution of the IVP

$$y' = f(t, y), \qquad y(t_0) = y_0 \,.$$

- Is it possible to extend the solution? (Under what conditions? continuation of the solution, out of our class topics)

# Existence and Uniqueness – linear ODEs

**Exercise 2.4.5**

Determine the interval on which the solution of the IVP is **certain** to exist.

$$(4 - t^2)y' + 2ty = 3t^2, \quad y(1) = -3$$

**Exercise 2.4.15**

Solve the IVP

$$y' + y^3 = 0, \quad y(0) = y_0$$

and determine how the interval where the solution exists depend on $y_0$.

$Link : Notes(B1.5)$

## Exercise 2.4.15

Solve the IVP

$$y' + y^3 = 0, \quad y(0) = y_0$$

and determine how the interval where the solution exists depend on $y_0$.

```
%% SAMPLE MATLAB CODE without INITIAL COND.
>>  syms y(t)
>> ode = diff(y,t) == -y^3
ode(t) =
diff(y(t), t) == -y(t)^3
>> dsolve(ode)
ans =
0
(2^(1/2)*(-1/(C2 - t))^(1/2))/2
-(2^(1/2)*(-1/(C2 - t))^(1/2))/2
```

# SAMPLE MATLAB CODE with INITIAL COND.

```
%% SAMPLE MATLAB CODE with INITIAL COND.
>> syms y(t)  Yo
>> ode = diff(y(t), t) == -y(t)^3
ode =
diff(y(t), t) == -y(t)^3
>> cond = y(0) == Yo;
>> dsolve(ode,cond)
ans =
(2^(1/2)*(1/(t + 1/(2*Yo^2)))^(1/2))/2
-(2^(1/2)*(1/(t + 1/(2*Yo^2)))^(1/2))/2
```

### Example 2.4.3

Solve the IVP : $y' = y^{1/3}, \quad y(0) = 0$

- Any theorem we've seen applies?
- Three solutions to the IVP :
  1. $y(x) = 0$
  2. $y(x) = \sqrt{\left(\frac{2}{3}x\right)^3}$
  3. $y(x) = -\sqrt{\left(\frac{2}{3}x\right)^3}$

# SAMPLE MATLAB with INITIAL CONDITION

**Example 2.4.3**

Solve the IVP : $y' = y^{1/3}, \quad y(0) = 0$

```
## SAMPLE MATLAB with INITIAL COND.
>> ode diff(y(t), t) == y(t)^(1/3)
ans =
diff(y(t), t) == y(t)^(1/3)
>> cond = y(0) == 0;
>> dsolve(ode,cond)
ans =
0
((2*t)/3)^(3/2)
```

- Notation : $q^{1/2} = \pm\sqrt{q}$