1. Write a few examples of
   (a) a homogeneous linear differential equation
   (b) a non-homogeneous linear differential equation
   (c) a linear and a non-linear differential equation.

2. Calculate $f'(t)$. Your answer can include other given functions and its derivatives if they are not explicitly defined (such as h).
   (a) $f(t) = t^2 g(t)$ where $g(t) = e^{t^2}$
   (b) $f(t) = h(v(t^2))$
   (c) $f(t) = \sqrt{(1 - g(t))}$ where $g(t) = sin(t)$
   (d) $f(t) = U(t, h(t))$ where $U_x = xy^2$ and $U_y = x^2 y$

3. Integrate with respect to x
   (a) $xsin(x)$
   (b) $x^2\sqrt{x^3+1}$
   (c) $\frac{3}{1-4x^2}$
   (d) $\frac{1+cos(x)}{sin(x)}$
   (e) $\frac{3}{1+4x^2}$, Hint: Recall $\frac{d}{dx}\left(tan^{-1}x\right) = \frac{1}{1+x^2}$

4. The determinant of the matrix $\begin{bmatrix} a & \alpha \\ b & \beta \end{bmatrix}$ is $a\beta - b\alpha$. Show that the determinant is 0 if and only if the vector $\begin{bmatrix} a \\ b \end{bmatrix}$ is constant multiple of the vector $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ (that means there is a constant C such that $a = C\alpha$ and $b = C\beta$).

5. Solve the following for a and b

$$\begin{aligned} 2a + 3b &= 0 \\ a + 5b &= 2 \end{aligned}$$

What is the relation between solving this linear equation and the matrix $\begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$ and its determinant.

6. Given a systems of differential equation

$$\begin{aligned} p(t)u'(t) + q(t)v'(t) &= f(t) \\ r(t)u'(t) + s(t)v'(t) &= g(t) \end{aligned}$$

for the unknown functions $u(t)$ and $v(t)$, suppose all other functions are known. Solve for u' and v'.

7. Consider the homogeneous differential equation

$$L[y] = 0 \tag{1}$$

where L is a **linear differential operator**, and assume all coefficients are differentiable functions.

(a) Consider the differential operator L_2 defined by

$$L_2[y] = L[L[y]]$$

is L_2 a linear operator?

(b) Which of the following are true? If true then show, if false then give a counter example.

i. If y_1 and y_2 are solutions then $y_1 - y_2$ solves (1).

ii. If y_1 and y_2 are solutions then $y_1 + Cy_2$ solves (1). (here C is a constant)

iii. If y_1 and y_2 are solutions then $y_1 \times y_2$ solves (1).

8. Let p, q be continuous functions, u, v be twice differentiable functions on $\mathbb{R}$. Define L by

$$L[y] = y'' + p(t)y' + q(t)y.$$

Which of the following are true? If true then show, if false give a counter example.

(a) $W(u, v)$ is a differentiable function (W is Wronskian)

(b) $W(u, v)$ can't achieve both negative and postive values

(c) If $L[u] = L[v] = 0$ then $W(u, v)$ can't achieve both negative and positive values

(d) If u is a constant multiple of v then $W(u, v) = 0$ for all t.

(e) Part (d) is true only if $L[u] = L[v] = 0$

9. Consider the differential equation

$$L[y] = f(x) \tag{2}$$

where L is a linear differential operator and assume all coefficients and f are differentiable functions. Which of the following are true? If true then show, if false then give a counter example.

   (a) If y_p is a solution of (2) then Cy_p solves (2).

   (b) If y_p is a solution of (2) and y_h is a solution to the associated homogeneous equation then $y_p + Cy_h$ solves (2).

   (c) If y_1 and y_2 are solutions of (2) then $y_1 - y_2$ solves (1).

   (d) If y_1 and y_2 are solutions of (2) then $y_1 + Cy_2$ solves (1) for any constant C.

10. Suppose $f(t)$ is a continuous function. Choose one of the following for each of the initial value problems below

   (a) has exactly one unique solution

   (b) has infinitely many solutions

   (c) no solutions

   (d) Can't conclude, depends on f

$$y'' + ty = f(t), \qquad y(0) = 1, \quad y'(0) = 2$$

$$y'' + ty = f(t), \qquad y(0) = 1$$

$$y' = f(t), \qquad y(0) = 0, \quad y'(0) = 1$$

11. Solve the following differential equations.

   (a) $\frac{y'}{(3x^2+1)} + y = 1, \qquad y(0) = 2$

   (b) $y'' - 6y' + 9y = 0, \qquad y(0) = 0, \qquad y'(0) = 1$

   (c) $\frac{y'}{\cos(x^2)} + 2xy^2 = 0, \qquad y(0) = 1$

12. Suppose population of a microorganism colony is modeled by the differential equation

$$\frac{dP}{dt} = -(P-1)(P-3)(P-7).$$

where numbers are written in the thousands. What would you expect the long term population of the colony ? How does it depend on the initial population ?

13. Consider the differential equation

$$y'' + P(x)y' + Q(x)y = 0 \qquad (DE)$$

where P and Q are both non-zero continuous functions on interval $I = (-1, 1)$. We don't know P, Q but we know that **at least 2** of the below functions

$$f(x) = x, \qquad g(x) = x^2, \qquad h(x) = x^2 - 4$$

solve the (DE) on I.

(a) Can all three be the solutions of (DE)? Why?

(b) Which are the solutions?

(c) Find $P(x)$

14. In addition to HW, some practice problems from the book :
§1.1: #1, 2, 26
§1.2: # 1(a), 2(a)
§2.1: #1, 9, 12
§2.2: #9, 11
§2.4: #3, 7, 11
§2.3: Solve the IVPs in Examples 2 and 3
§2.3: #9, 10
§2.5: #9, 23
§2.6: #23, 26, 30, 31
§3.1: #2, 4
§3.2: #2, 4, 14, 25
§3.3: #8, 27
§3.4: #2, 16, 24
§3.5: #5, 19
§3.6: #9, 13
§6.1: #3, 4, 11, 22
§6.2: #22, 33
§6.3: #20, 24
§6.5: #9, 15
§6.6: #6, 11, 14

15. For past year exams see: https://www.math.dartmouth.edu/~m23s18/lectureNotes/

Below are some selected questions from last years final exam.

16. *[20 points]* TRUE or FALSE? You must provide a *concise* justification if you claim the statement is false. In some cases a counter-example might be the easiest way to justify your answer.

    (a) If $y_1(t)$ and $y_2(t)$ are both solutions to $y''(t)+p(t)y'(t)+q(t)y(t)=g(t)$, where p, q and g are continuous for all t, then so is $c_1y_1(t)+c_2y_2(t)$.

    (b) The appropriate form for a particular solution of $y''+6y'+13y=e^{-3t}\cos(2t)$ is $Y(t)=e^{-3t}(A\cos(2t)+B\sin(2t))$.

    (c) Suppose y_1 and y_2 are both solutions to $y''(t)+p(t)y'(t)+q(t)y(t)=g(t)$, where p, q and g are continuous on an open interval I. If y_1 and y_2 are both zero at the same point in I, then they cannot be a fundamental set of solutions on that interval.

17. *[16-points]* Consider the homogeneous equation for $t>0$:

$$t^2y''+ty'+y=0.$$

    (a) (3 points) Verify that $y_1(t)=\cos(\ln t)$ and $y_2(t)=\sin(\ln t)$ form a fundamental set of solutions for this problem.

    (b) (13 points) Find the general solution to

$$t^2y''+ty'+y=\ln t.$$

    Note that you will get **ZERO** credit if you simply guess a solution that works. Hint: $\int x\cos x dx = x\sin x+\cos x+C$ and $\int x\sin x dx=\sin x - x\cos x + C$.

18. *[22- points]* Consider the initial value problem for $y(t)$:

$$y''+8y'+16y=0$$

with initial conditions $y(0)=1$ and $y'(0)=\alpha$. Assume that $t>0$.

    (a) (8 points) Find the solution.

    (b) (3 points) For what values of α is the solution always positive?

    (c) (5 points) Sketch the solution for $\alpha=-5$ and discuss its short and long term behavior.