Math 23, Spring 2018

Dartmouth College

Week 9

Recall

$$y'' + \lambda y = 0, \quad y(0) = 0, \; y(\pi) = 0$$

has eigenvalues $\lambda_n = n^2$ and corresponding eigenfunctions $y_n(t) = sin(nx)$ for $n = 1, 2, 3, \ldots$.

Similarly,

$$y'' + \lambda y = 0, \quad y(0) = 0, \; y(L) = 0 \qquad (BVP)$$

has eigenvalues $\lambda_n = n^2\pi^2/L^2$ and corresponding eigenfunctions $y_n(t) = sin(n\pi x/L)$ for $n = 1, 2, 3, \ldots$.

Fourier Series of $f(x)$

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m cos\left(\frac{m\pi x}{L}\right) + b_m sin\left(\frac{m\pi x}{L}\right)\right)$$

has fundamental period $2L$.

Orhogonality of cosine and sine

The set

$$F = \left\{ \frac{1}{\sqrt{2}}, cos\left(\frac{\pi x}{L}\right), cos\left(\frac{2\pi x}{L}\right), cos\left(\frac{3\pi x}{L}\right), \ldots, sin\left(\frac{\pi x}{L}\right), sin\left(\frac{2\pi x}{L}\right), sin\left(\frac{3\pi x}{L}\right), \ldots \right\}$$

is an orthogonal set on interval $(-L, L)$ that satisfies:

$$\langle u, v \rangle = 0, \qquad if\ u \neq v$$
$$\langle u, v \rangle = L, \qquad if\ u = v$$

Link: Notes (B 9.1)
- Inner product of functions and orthogonality of F

Fourier Coefficients

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m cos\left(\frac{m\pi x}{L}\right) + b_m sin\left(\frac{m\pi x}{L}\right)\right)$$

- $a_0 = \frac{1}{L}\langle 1, f\rangle = \frac{1}{L}\int_{-L}^{L} f(x)dx$
- $a_n = \frac{1}{L}\left\langle cos\left(\frac{n\pi x}{L}\right), f\right\rangle = \frac{1}{L}\int_{-L}^{L} f(x)cos\left(\frac{n\pi x}{L}\right) dx$
- $b_n = \frac{1}{L}\left\langle sin\left(\frac{n\pi x}{L}\right), f\right\rangle = \frac{1}{L}\int_{-L}^{L} f(x)sin\left(\frac{n\pi x}{L}\right) dx$

- Thus given $f(x)$ we can calculate its FS (assuming it is convergent)

Link: Notes (B 9.1)
- Euler-Fourier formula's

Fourier Convergence Theorem

$f(x)$ is **piecewise continuous** if it is continuous except a finite number of points.

Theorem. Suppose f, f' are piecewise continuous on $(-L, L)$. Then $f(x)$ has Fourier series expansion

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m cos\left(\frac{m\pi x}{L}\right) + b_m sin\left(\frac{m\pi x}{L}\right)\right) \qquad (FS)$$

where a_i, b_i are given as above such that FS converges to $f(x)$ at all points x where f is continuous, and converges to $\frac{f(x+)+f(x-)}{2}$ where $f(x)$ is discontinuous.

Notes:

- We can always extend f to a periodic function on $\mathbb{R}$ with period $2L$.
- Other domains such as (α, β) can always be "shifted" to $(-L, L)$