Math 23, Spring 2018

Dartmouth College

Week 8

Boundary Value Problems

BVP (Boundary Value Problem) :

$$\begin{aligned} y'' + p(t)y' + q(t)y =& g(t) \\ y(\alpha) =& y_0 \\ y(\beta) =& y_1 \end{aligned}$$

If g, y_0, y_1 are all 0 then **homogeneous** otherwise **non-homogeneous**

Boundary Value Problems

Recall. Linear homogeneous and non-homogeneous systems

$$A\boldsymbol{x} = \mathbf{0}, \qquad (H), \qquad\qquad A\boldsymbol{x} = \boldsymbol{b} \qquad (NH)$$

- $\mathbf{0}$ is always a solution of (H)
- $\mathbf{0}$ is the only solution of (H) $\iff$ (NH) has a unique solution
- (H) has a non-zero solution $\iff$ (NH) has either no solution or infinitely many

Note: We've observed the same for the linear DE, for example second order constant coefficient IVPs

For sample BVPs

4 BVPs :

- $y'' + 2y = 0, \quad y(0) = 1,\ y(\pi) = 0$: Non-homogeneous. Has unique solution
- $y'' + y = 0, \quad y(0) = 1,\ y(\pi) = a$: Infinitely many soln if $a = -1$, no soln. if $a \neq -1$
- $y'' + 2y = 0, \quad y(0) = 0,\ y(\pi) = 0$: Homogeneous. $y = 0$ is unique solution
- $y'' + y = 0, \quad y(0) = 0,\ y(\pi) = 0$: Homogeneous. Has a non-zero solution

Note: How to solve these?

Eigenvalue and Eigenfunctions

$(A - \lambda I)\boldsymbol{x} = 0$ considering $(D^2 + \lambda)y = 0$

For the BVP

$$y'' + \lambda y = 0, \quad y(0) = 0,\, y(\pi) = 0 \qquad (BVP)$$

the values of λ for which (BVP) has a non-zero solution are called **eigenvalues**, and the corresponding non-zero solution $y(t)$ is called **eigenfunction** .

Link: Notes (B 8.3)
- Example.
- Finding eigenvalues and eigenfunctions

Fourier Series

Consider

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m cos\left(\frac{m\pi x}{L}\right) + b_m sin\left(\frac{m\pi x}{L}\right) \right)$$

- converges?
- if converges what functions has such representations

Link: Notes (B 8.3)
- Periodic functions and Fourier series