Math 23, Spring 2018

Dartmouth College

Week 5

Main Theorem

Theorem 6.2.1

1. If f is continuous and f' is piecewise continuous on $[0, A]$, for any $A > 0$
2. If $|f(t)| \leq Ke^{at}$ for $t > M$, with $K, M, a \in \mathbb{R}$ and $K, M > 0$.

Then the Laplace transform $\mathcal{L}(f')(s)$ exists for $s > a$ and

$$\mathcal{L}\left(f'\right)(s) = s\mathcal{L}(f)(s) - f(0)$$

Corollary

Corollary 6.2.2

1. If f, f', ..., $f^{(n-1)}$ are continuous on $[0, A]$, for any $A > 0$
2. If $|f^{(i)}(t)| \leq Ke^{at}$ for $t > M$ and $i = 0, \ldots, n-1$, with $K, M, a \in \mathbb{R}$ and $K, M > 0$.

Then the Laplace transform $\mathcal{L}\left(f^{(n)}\right)(s)$ exists for $s > a$ and

$$\mathcal{L}\left(f^{(n)}\right)(s) = s^n \mathcal{L}(f)(s) - s^{n-1} f(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0)$$

Exercise

$$\mathcal{L}\left(f^{(n)}\right)(s) = s^n\mathcal{L}(f)(s) - s^{n-1}f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

Exercise 6.2.11

Use the Laplace transform to solve

$$y'' - y' - 6y = 0; \quad y(0) = 1, y'(0) = -1$$

Link: Notes (B 5.1)

6.3 Step functions

Definition

The function $u_c(t) := \begin{cases} 0, & t < c \\ 1, & t \geq c. \end{cases}$ is known as the **unit step function** or **Heaviside function**.

Exercise

Check $\mathcal{L}(u_c)(s) = \begin{cases} e^{-cs}\frac{1}{s}, & c > 0 \\ \frac{1}{s} & c < 0 \end{cases} \quad s > 0$

Shifting $\iff$ multiplying with exp

$$u_c(t) := \begin{cases} 0, & t < c \\ 1, & t \geq c. \end{cases}$$

Theorem 6.3.1

If $\mathcal{L}(f)(s)$ exists for $s > a \geq 0$ and $c > 0$, then

$$\mathcal{L}\left[u_c(t)f(t-c)\right](s) = e^{-cs}\mathcal{L}(f)(s), \quad s > a$$

Theorem 6.3.2

If $\mathcal{L}(f)(s)$ exists for $s > a \geq 0$, then

$$\mathcal{L}\left[e^{ct}f(t)\right](s) = \mathcal{L}(f)(s-c), \quad s > a + c$$

Exercise 6.3.20

Exercise

Find the inverse Laplace Transform of

$$\frac{e^{-2s}}{s^2+s-2}$$

- $\displaystyle \frac{1}{s^2+s-2} = \frac{1}{3}\left(\frac{1}{s-1} - \frac{1}{s+2}\right)$
- $\displaystyle \frac{1}{s-a} = \mathcal{L}(e^{at})$
- $e^{-2s}\mathcal{L}(f)(s) = \mathcal{L}(u_2(t)f(t-2))$
- $\mathcal{L}^{-1}\left(\frac{e^{-2s}}{s^2+s-2}\right) = \frac{1}{3}u_2(t)(e^{t-2} - e^{4-2t})$

Exercise 6.2.24

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Solve $y'' + 4y = \begin{cases} 1, & 0 \le t < \pi, \\ 0, & t \ge \pi; \end{cases} \quad y(0) = 1, \quad y'(0) = 0$

Note: $y'' + 4y = 1 - u_\pi(t)$

Link: Notes (B 5.2)