Math 23, Spring 2018

Dartmouth College

Week 4

Nonhomogeneous equations

We now considering

$$y'' + p(t)y' + q(t)y = g(t)$$

Main theorem

Theorem 3.5.1

If $\tilde{y}_1$ and $\tilde{y}_2$ are solutions of

$$y'' + p(t)y' + q(t)y = g(t), \tag{NH}$$

then $\tilde{y}_1 - \tilde{y}_2$ is a solution of

$$y'' + p(t)y' + q(t)y = 0 \tag{H}$$

Corollary

Assume that $\{y_1, y_2\}$ is a fundamental solution set of (H), and y_p is **a particular solution** of (NH). Then general solution of (NH) can be written as

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + y_p(t).$$

$\boxed{Link : Notes(B4.1)}$

General solutions of nonhomogeneous equations

To find the general solution for the nonhomogeneous equation

$$y'' + p(t)y' + q(t)y = g(t), \tag{NH}$$

we need to find **one** solution for it, y_p known as a **particular solution**, then find a fundamental solution set $\{y_1, y_2\}$ for the homogeneous equation

$$y'' + p(t)y' + q(t)y = 0. \tag{H}$$

Then, we can write the general solution for (NH) as

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + y_p(t).$$

Above fact is useful for constant coefficient DE's :
since we already know the general solution of the homogeneous problem. All we need is to find a particular solution to the non-homogeneous problem.

Method of undetermined coefficients

How to to find a particular solution for the constant coefficient non-homogeneous equation?

Exercise

For each of the below, write a set S of functions such that linear combinations of the functions in S contains all derivatives of the given function

1. $sin(\alpha t)$
2. $cos(\alpha t)$
3. $t^3 + t$
4. e^{7t}
5. e^{-3t}
6. $t^2 e^{-3t}$

Example: $y'' + y' + y = g(t)$

$$y'' + y' + y = g(t)$$

- Homogeneous equation: $y'' + y' + y = 0$
- Characteristic equation: $r^2 + r + 1 = 0 \Rightarrow r = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$
- General solution for the homogeneous equation :

$$y(t) = c_1 e^{-t/2} \sin\left(\frac{\sqrt{3}}{2} t\right) + c_2 e^{-t/2} \cos\left(\frac{\sqrt{3}}{2} t\right)$$

- A general solution for $y'' + y' + y = g(t)$ will be of the shape:

$$y(t) = c_1 e^{-t/2} \sin\left(\frac{\sqrt{3}}{2} t\right) + c_2 e^{-t/2} \cos\left(\frac{\sqrt{3}}{2} t\right) + y_p(t)$$

where $y_p(t)$ **depends** on $g(t)$.

Example: $y'' + y' + y = t^3 + 1$

- $g(t) = t^3 + 1 \rightsquigarrow y_p(t) = t^s(A_3t^3 + A_2t^2 + A_1t + A_0)$ for some A_3, A_2, A_1, A_0
- No duplication $s = 0$
- Thus $y_p(t)$ is of the shape $A_3t^3 + A_2t^2 + A_1t + A_0$
- $y_p'' + y_p' + y_p = t^3 + 1 \Rightarrow$

$$(6A_3t + 2A_2) + (3A_3t^2 + 2A_2t + A_1) + (A_3t^3 + A_2t^2 + A_1t + A_0) = t^3 + 1$$

$$\Leftrightarrow$$

$$A_3t^3 + (3A_3 + A_2)t^2 + (6A_3 + 2A_2 + A_1)t + (2A_2 + A_1 + A_0) = 1t^3 + 0t^2 + 0t + 1$$

$$\Leftrightarrow \begin{cases} 1 &= A_3 \\ 0 &= 3A_3 + A_2 \\ 0 &= 6A_3 + 2A_2 + A_1 \\ 1 &= 2A_2 + A_1 + A_0 \end{cases} \Leftrightarrow \begin{cases} A_3 &= 1 \\ A_2 &= -3 \\ A_1 &= 0 \\ A_0 &= 7 \end{cases}$$

Example: $y'' + y' + y = t^3 + 1$

Answer: A general solution for

$$y'' + y' + y = t^3 + 1$$

is

$$y = \underbrace{c_1 e^{-t/2} \sin\left(\frac{\sqrt{3}}{2}t\right) + c_2 e^{-t/2} \cos\left(\frac{\sqrt{3}}{2}t\right)}_{\text{the general solution for the homogeneous equation}} + \underbrace{t^3 - 3t^2 + 7}_{\text{a particular solution}}$$

Example: $y'' + 4y' + 3y = g(t)$

- $r^2 + 4r + 3 = (r+1)(r+3)$, thus the general homogeneous solution is
$$y_H(t) = c_1 e^{-3t} + c_2 e^{-t}$$
- $y'' + 4y' + 3y = e^{-3t}(t+3)$
Candidate: $t^s e^{-3t}(A_1 t + A_0) \rightsquigarrow s = 1$
$$y_p(t) = e^{-3t} t(A_1 t + A_0)$$
- $y'' + 4y' + 3y = e^{4t}(t+3)$
Candidate: $t^s e^{4t}(A_1 t + A_0) \rightsquigarrow s = 0$
$$y_p(t) = e^{4t}(A_1 t + A_0)$$
- $y'' + 4y' + 3y = \cos(2t)$
Candidate: $t^s(\cos(2t)A_0 + \sin(2t)B_0) \rightsquigarrow s = 0$ and
$$y_p(t) = \cos(2t)A_0 + \sin(2t)B_0$$

Some particular solutions for $c_n y^{(n)} + \cdots + c_1 y' + c_0 y = g(t) \quad (\Delta)$

Write $p(r) := c_n r^n + \cdots + c_1 r + c_0$

$g(t)$ — RHS of (Δ)	$y_p(t)$ — a particular solution for (Δ)
$a_k t^k + \cdots + a_1 t + a_0$	$t^s(A_k t^k + \cdots + A_1 t + A_0)$
$e^{\alpha t}(a_k t^k + \cdots + a_1 t + a_0)$	$t^s e^{\alpha t}(A_k t^k + \cdots + A_1 t + A_0)$
$e^{\alpha t}\cos(\beta t)(a_k t^k + \cdots + a_1 t + a_0)$ + $e^{\alpha t}\sin(\beta t)(b_k t^k + \cdots + b_1 t + b_0)$	$e^{\alpha t} t^s \cos(\beta t)(A_k t^k + \cdots + A_1 t + A_0)$ + $e^{\alpha t} t^s \sin(\beta t)(B_k t^k + \cdots + B_1 t + B_0)$
Note: we can have a_i or $b_i = 0$	

where s is the smallest integer to ensure there is no duplication (i.e. no term of y_p is a solution of homogeneous problem)

$y'' + y' + y = e^{-t/2}(t^2 + 3)$

- $g(t) = e^{-t/2}(t^2+3) \rightsquigarrow y_p(t) = e^{-t/2}t^s(A_2t^2 + A_1t + A_0)$ for some A_2, A_1, A_0
- No duplication: $s = 0$
- Thus $y_p(t)$ is of the shape $e^{-t/2}(A_2t^2 + A_1t + A_0)$
- Plugging it all in and solving the system we get

$$A_0 = \frac{4}{9}, \quad A_1 = 0, \quad A_2 = \frac{12}{9}$$

- Thus the general solution for $y'' + y' + y = e^{-t/2}(t^2+3)$ is

$$y(t) = \underbrace{c_1 e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right) + c_2 e^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t\right)}_{\text{the general solution for the homogeneous equation}} + \underbrace{e^{-t/2}\frac{12t^2+4}{9}}_{\text{a particular solution}}$$

Example: $y'' + y' + y = \cos(\sqrt{3}t)t^2$

$$y'' + y' + y = \cos(\sqrt{3}t)t^2$$

- No duplication: $s = 0$

$$y_p(t) = \cos(\sqrt{3}t)(A_2 t^2 + A_1 t + A_0) + \sin(\sqrt{3}t)(B_2 t^2 + B_1 t + B_0)$$

- a system in six unknowns and six equations...

$$A_2 = \frac{49}{343}\sqrt{3} \qquad B_2 = \frac{-98}{343}$$

$$A_1 = \frac{84}{343}\sqrt{3} \qquad B_1 = \frac{322}{343}$$

$$A_0 = \frac{-222}{343}\sqrt{3} \qquad B_0 = \frac{-18}{343}$$

Example: $y'' + y' + y = e^{-t/2}\cos(\sqrt{3}t/2)(t+1)$

$$y'' + y' + y = e^{\frac{-t}{2}} \cos\left(\frac{\sqrt{3}t}{2}\right)(t+1)$$

- Candidate y_p

$$y_p(t) = t^s e^{-t/2} \sin\left(\frac{\sqrt{3}}{2}t\right)(A_1 t + A_0) + t^s e^{-t/2} \cos\left(\frac{\sqrt{3}}{2}t\right)(B_1 t + B_0)$$

- Duplication. Smallest value to guarantee no duplication is $s = 1$.

$$y_p(t) = te^{-t/2} \sin\left(\frac{\sqrt{3}}{2}t\right)(A_1 t + A_0) + te^{-t/2} \cos\left(\frac{\sqrt{3}}{2}t\right)(B_1 t + B_0)$$

Example: $y'' - 6y' + 9y = g(t)$

- $r^2 - 6r + 9 = (r-3)^2$, thus the general homogeneous solution is

$$y_H(t) = c_1 e^{3t} + c_2 t e^{3t}$$

- $y'' - 6y' + 9y = e^{3t}(t+2)$
  candidate :

$$y_p(t) = t^s e^{3t}(A_1 t + A_0)$$

  $\rightsquigarrow s = 2$ and

$$y_p(t) = e^{3t} t^2 (A_1 t + A_0)$$

- $y'' - 6y' + 9y = e^{3t}\cos(t)(t+2)$
  $\rightsquigarrow s = 0$ and

$$y_p(t) = e^{3t}\cos(t)(A_1 t + A_0) + e^{3t}\sin(t)(B_1 t + B_0)$$

Example: $y'' - 6y' + 9y = g_1(t) + g_2(t)$

$$y'' - 6y' + 9y = t + 1 + 3\cos(2t)$$

$$g_1(t) = t + 1$$

and

$$g_2(t) = 3\cos(2t)$$

Solve $L[y] = g_1$ and $L[y] = g_2$ separately then add. Why does it work?

Superposition of particular solutions

Theorem

Let $L[y] = y'' + p(t)y' + q(t)y$. If y_1 is a solution of

$$L[y] = g_1(t)$$

and y_2 is a solution of

$$L[y] = g_2(t),$$

then $y_1 + y_2$ is a solution of

$$L[y] = g_1(t) + g_2(t).$$

Example: $y'' - 6y' + 9y = g(t)$

- $y'' - 6y' + 9y = t + 1 \rightsquigarrow$
$$y_p(t) = \frac{1}{9}t + \frac{5}{27}$$
- $y'' - 6y' + 9y = 3\cos(2t) \rightsquigarrow$
$$y_p(t) = \frac{15}{169}\cos(2t) + \frac{-36}{169}\sin(2t)$$
- $y'' - 6y' + 9y = t + 1 + 3\cos(2t) \rightsquigarrow$

$$y_p(t) = \underbrace{\frac{1}{9}t + \frac{5}{27}}_{\text{particular solution for } g(t)=t+1} + \underbrace{\frac{15}{169}\cos(2t) + \frac{-36}{169}\sin(2t)}_{\text{particular solution for } g(t)=3\cos(2t)}$$

Variation of parameters

Link: Notes (B 4.2)
 Variation of parameters

The Laplace Transform

Link: Notes (B 4.3)

Improper integrals

Piecewise continuous functions and their integrals

Integral transforms

Definition

$$\mathcal{L}(f)(s) := \int_0^{+\infty} e^{-st} f(t)\,\mathrm{d}t \quad \text{(if the integral converges)}$$

The Laplace Transform

Definition

$$\mathcal{L}(f)(s) := \int_0^{+\infty} e^{-st} f(t)\,\mathrm{d}t \quad \text{(if the integral converges)}$$

Notes :

(1) $\mathcal{L}$ maps a function to a function : $f \mapsto \mathcal{L}(f)$

Here f is a function in t domain, $\mathcal{L}(f)$ is a function in s domain.

$$\left\{ \begin{array}{rcl} f : [0,+\infty) & \longrightarrow & \mathbb{R} \\ t & \longmapsto & f(t) \end{array} \right\} \longmapsto \left\{ \begin{array}{rcl} \mathcal{L}(f) : I & \longrightarrow & \mathbb{R} \\ s & \longmapsto & \mathcal{L}(f)(s) \end{array} \right\}$$

(2) $\mathcal{L}$ is a linear operator:

$$\mathcal{L}(c_1 f_1 + c_2 f_2)(s) = c_1 \mathcal{L}(f_1)(s) + c_2 \mathcal{L}(f_2)(s)$$

(3) Its power: $\mathcal{L}$ is invertable and

$\mathcal{L}$: DE in t-domain $\longmapsto$ algebraic equations in the s-domain.

Existence

- $\mathcal{L}(1) = \frac{1}{s}$, $s > 0$
- $\mathcal{L}(e^{at}) = \frac{1}{s-a}, \quad s > a$

Theorem 6.1.2

1. If f is piecewise continuous on $[0, A]$, for any $A > 0$
2. If $|f(t)| \leq Ke^{at}$ for $t > M$, with $K, M, a \in \mathbb{R}$ and $K, M > 0$.

Then the Laplace transform $\mathcal{L}(f)(s)$ exists for $s > a$.

Ex: Write a function whose Laplace transform doesn't exist anywhere.

More examples

- $\mathcal{L}(\cos(\beta t)) = ?$
  $\mathcal{L}(\sin(\beta t)) = ?$
- Calculate using Euler's identity and linearity of $\mathcal{L}$:

$$e^{(\alpha+i\beta)t} = e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))$$

$$\left|e^{(\alpha+i\beta)t}\right| = e^{\alpha t}\sqrt{\cos(\beta t)^2 + \sin(\beta t)^2} = e^{\alpha t}$$

$$\mathcal{L}\left(e^{(\alpha+i\beta)t}\right) = \frac{1}{s-(\alpha+\beta i)}, \quad s > \alpha$$

Exercise

Deduce $\mathcal{L}\left(e^{\alpha t}\cos(\beta t)\right)$ and $\mathcal{L}\left(e^{\alpha t}\sin(\beta t)\right)$ with $\alpha, \beta \in \mathbb{R}$.