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Recall: Homogeneous equations with constant coefficients

Definition

To the equation

$$ay'' + by' + cy = 0 \quad a, b, c\mathbb{R}$$

we associate a **characteristic equation**

$$ar^2 + br + c = 0.$$

If the characteristic equation has

1. two different real roots: $r_1, r_2 \in \mathbb{R} \Rightarrow y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ **Done! ✓**
2. a double root: $r \in \mathbb{R} \Rightarrow y = c_1 e^{rt} + c_2 t e^{rt}$ **Why?**
3. two complex roots: $\alpha \pm i\beta \in \mathbb{C} \Rightarrow y = c_1 e^{\alpha t}\cos(\beta t) + c_2 e^{\alpha t}\sin(\beta t)$ **Why?**

Euler's identity

$$e^{\alpha+i\beta} = e^{\alpha}(\cos\beta + i\sin\beta), \quad \alpha,\beta \in \mathbb{R}$$

Claim

$$\frac{\mathrm{d}}{\mathrm{d}t} e^{(\alpha+i\beta)t} = (\alpha+i\beta)e^{(\alpha+i\beta)t}$$

- If $r = \alpha + i\beta$ is a root, then $e^{(\alpha+i\beta)t}$ also satisfies the differential equation. (we only need $(e^{rt})' = re^{rt}$)
- If $r = \alpha + i\beta$ is a root, then $\alpha - i\beta$ is also a root!

Complex roots

If $\alpha \pm i\beta$ are roots to the characteristic equation, then we have two **complex** solutions $e^{(\alpha \pm i\beta)t}$.

We would like two **real** solutions.

How do we get $y = c_1 e^{\alpha t}\cos(\beta t) + c_2 e^{\alpha t}\sin(\beta t)$?

$$\widetilde{y_1} = e^{(\alpha + i\beta)t} = e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))$$
$$\widetilde{y_2} = e^{(\alpha - i\beta)t} = e^{\alpha t}(\cos(\beta t) - i\sin(\beta t))$$

$$y_1 = \frac{\widetilde{y_1} + \tilde{y_2}}{2} = ?$$
$$y_2 = \frac{\widetilde{y_1} - \tilde{y_2}}{2i} = ?$$

Complex roots

$$\widetilde{y_1} = e^{(\alpha+i\beta)t} = e^{\alpha t}(\cos(\beta t) + i\sin(\beta t)) \quad \widetilde{y_2} \quad = e^{(\alpha-i\beta)t} = e^{\alpha t}(\cos(\beta t) - i\sin(\beta t))$$

$$y_1 = \frac{\widetilde{y_1} + \tilde{y_2}}{2} = e^{\alpha t}\cos\beta t \qquad (= \text{Real}(\tilde{y}_1))$$

$$y_2 = \frac{\widetilde{y_1} - \tilde{y_2}}{2i} = e^{\alpha t}\sin\beta t \qquad (= \text{Imag}(\tilde{y}_1))$$

These are also two solutions, but now real!

Q: Do they form a fundamental solution set?

A: Yes, we already checked that $W(\widetilde{y_1}, \tilde{y_2})(t) \neq 0$ as long $r_1 = \alpha + i\beta \neq \alpha - i\beta = r_2$ and

$$W(y_1, y_2)(t) = \frac{-2}{4i} W(\widetilde{y_1}, \tilde{y_2})(t) \quad \text{check it!}$$

Exercise 18

Solve the IVP: $\begin{cases} y'' + 4y' + 5y = 0 \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$

1. $r^2 + 4r + 5 = 0 \rightsquigarrow r = \frac{-4 \pm \sqrt{16-20}}{2} = -2 \pm i$
2. $y_1(t) = e^{-2t}\cos(t)$
   $y_2(t) = e^{-2t}\sin(t)$
   $y(t) = c_1 e^{-2t}\cos(t) + c_2 e^{-2t}\sin(t)$
3. Solve for c_1 and c_2
4. $y(t) = e^{-2t}\cos(t) + 2e^{-2t}\sin(t)$
5. What happens as $t \to +\infty$

What about multiple roots?

If the characteristic equation $ar^2 + br + c = 0$. has a **double root** $r \in \mathbb{R}$, then

$$y_1 = e^{rt}$$
$$y_2 = te^{rt}$$

Where did did y_2 come from?

- Guessing? Only works some times.
- A: Using y_1 we will reduce the order of our ODE.

Reduction of order method

Let y_1 be a solution to

$$y'' + p(t)y' + q(t)y = 0$$

Let's search for $y_2(t) = v(t)y_1(t)$ with $v(t)$ arbitrary.

Plugin y_2, y_2' and y_2'' in the original ODE. What do you get?

$$v''y_1 + v'(2y_1' + py_1) = 0$$

$$u'y_1 + u(2y_1' + py_1) = 0, \quad u = v'$$

and we know how to solve for u, and $v = \int u$

$$y_2 = vy_1, \text{ where } u' = v \text{ and } u'y_1 + u(2y_1' + py_1) = 0$$

Example

Solve $y'' - 2ay' + a^2y = 0$

Exercise 3.4.24

Solve $t^2y'' + 2ty' - 2y = 0$, given $y_1 = t$

Exercise 16

a) Find a solution to the initial value problem as a function of b

$$y'' - y' + \frac{1}{4}y = 0, \quad y(0) = 2, \quad y'(0) = b$$

b) Determine a critical value of b that separates solutions that grow positively from those that eventually grow negatively.

$$y(t) = e^{t/2}(bt - t + 2)$$
$$y'(t) = \frac{1}{2}e^{t/2}((b-1)t + 2b)$$