Math 23, Spring 2017

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Recall: the Wronkskian

Definition

Let y_1 and y_2 be two solutions of the homogeneous ODE

$$y'' + p(t)y' + q(t)y = 0.$$

The **Wronkskian** of these solutions is a function in t

$$W(y_1, y_2)(t) := \det \begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix} = y_1(t)y_2'(t) - y_2(t)y_1'(t)$$

Recall: Theorem 3.2.3

Theorem 3.2.3

Let $y_1(t)$ and $y_2(t)$ be two solutions of the homogeneous ODE $y'' + p(t)y' + q(t)y = 0$, that are defined at t_0. If $W(y_1, y_2)(t_0) \neq 0$, then **every** solution to the initial value problem

$$y(t_0) = y_0, \quad y'(t_0) = y'_0$$

can solved in the form

$$y(t) = C_1 y_1(t) + C_2 y_2(t).$$

What is the key idea?

Fundamental Solution set

Definition

Assume that $p(t)$ and $q(t)$ are **continuous** in (α, β). Further, assume that $y_1(t)$ and $y_2(t)$ are two solutions in (α, β) for

$$y'' + p(t)y' + q(t)y = 0.$$

If $W(y_1, y_2)(t_0) \neq 0$ for **some** $t_0 \in (\alpha, \beta)$ then $\{y_1(t), y_2(t)\}$ is called the **fundamental solution set** of the ODE on the interval (α, β).

Q: Why do we call them **fundamental**?

A: Because every solution in (α, β) can be written as $y(t) = C_1 y_1(t) + C_2 y_2(t)$

Even if the initial conditions are given at $t_1 \neq t_0$, with $t_1 \in (\alpha, \beta)$

Theorem 3.2.4

Theorem 3.2.4

Suppose that $\{y_1(t), y_2(t)\}$ is a fundamental solution set for

$$y'' + p(t)y' + q(t)y = 0,$$

on the interval (α, β) (i.e., p, q continuous in (α, β) and $W(y_1, y_2)(t_0) \neq 0$ for some $t_0 \in (\alpha, \beta)$).
Then, **every** solution in (α, β) can be written as

$$y(t) = C_1 y_1(t) + C_2 y_2(t).$$

Even if the initial conditions are given at $t_1 \neq t_0$, with $t_1 \in (\alpha, \beta)$

Corollary

$W(y_1, y_2)(t) \neq 0$ for $t \in (\alpha, \beta)$

Abel's Theorem (3.2.7)

If y_1, y_2 are solutions on the interval (α, β) of

$$y'' + p(t)y' + q(t)y = 0,$$

and $p(t), q(t)$ are continuous on (α, β).
Then,

$$W(y_1, y_2)(t) = Ce^{-\int p(t)\,\mathrm{d}t} \quad \text{for } t \in (\alpha, \beta)$$

Proof: $W(y_1, y_2)(t)$ satisfies the differential equation $W' + p(t)W = 0$.

Remarks:

- $e^{-\int p(t)\,\mathrm{d}t}$ is never zero.
- $W(y_1, y_2)(t)$ is either zero at all points or nonzero at all points in (α, β).
- If nonzero, then every IVP has a unique solution of the form $c_1 y_1 + c_2 y_2$

How to find a general solution for for

$$y'' + p(t)y' + q(t)y = 0$$

with p, q continuous on (α, β).

1. Find **two** solutions for which $W(y_1, y_2)(t_0) \neq 0$ for some $t_0 \in (\alpha, \beta)$.
2. **Every** solution is of the form $y(t) = c_1 y_1(t) + c_2 y_2(t) \quad c_1, c_2 \in \mathbb{R}$

Problem 7 Midterm 1, Winter 2014

a) Find two constants n such that $y = t^n$ is a solution to the differential equation

$$t^2 y'' + 3ty' - 3y = 0$$

b) Write down a general solution for $t < 0$. Did you justify everything? $n = -3, 1$.

Homogeneous equations with constant coefficients

Definition

To the equation

$$ay'' + by' + cy = 0 \quad a, b, c\mathbb{R}$$

we associate a **characteristic equation**

$$ar^2 + br + c = 0.$$

If the characteristic equation has

1. **two different real roots** $r_1, r_2 \in \mathbb{R} \Rightarrow y = C_1 e^{r_1 t} + C_2 e^{r_2 t},\ C_1, C_2 \in \mathbb{R}$ Done! ✓
2. **double root** $r \in \mathbb{R} \Rightarrow y = C_1 e^{rt} + C_2 t e^{rt},\ C_1, C_2 \in \mathbb{R}$ **Why?**
3. **two complex roots** $\alpha \pm i\beta \Rightarrow y = C_1 e^{\alpha t} \cos(\beta t) + C_2 e^{\alpha t} \sin(\beta t)\ C_1, C_2 \in \mathbb{R}$ **Why?**

Next time: Euler's identity

$$e^{\alpha+i\beta} = e^{\alpha}(\cos\beta + i\sin\beta), \quad \alpha, \beta \in \mathbb{R}$$

Claim

$$\frac{\mathrm{d}}{\mathrm{d}t}e^{(\alpha+i\beta)t} = (\alpha+i\beta)e^{(\alpha+i\beta)t}$$

- If $r = \alpha + i\beta$ is a root, then $e^{(\alpha+i\beta)t}$ also satisfies the differential equation. (Before we only used $(e^{rt})' = re^{rt}$)
- If $r = \alpha + i\beta$ is a root, then $\alpha - i\beta$ is also a root!