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Second order linear equations

- **Second order** equation

$$\Rightarrow y'' = f(t, y,')$$

- Second order **linear equation**

$$\Rightarrow y'' + p(t)y' + q(t)y = g(t)$$

Homogeneous equations and initial conditions

Definition

A second order linear equation

$$y'' + p(t)y' + q(t)y = g(t)\,.$$

is called **homogeneous** if $g(t) = 0$.

Otherwise it is called **nonhomogeneous**.

Note that when specifying an IVP for a second-order equation, we have to give **two** initial conditions, both the value of the y and y':

$$y(t_0) = y_0, \qquad y'(t_0) = y'_0\,.$$

For example, to solve $y'' = f(t)$ you will need to integrate the $f(t)$ and $\int_{t_0}^{t} f(s)\,\mathrm{d}s$, thus you will need to deal with two arbitrary constants.

Existence and uniqueness

Theorem 3.2.1

Consider the IVP

$$y'' + p(t)y' + q(t)y = g(t), \qquad y(t_0) = y_0, \qquad y'(t_0) = y'_0$$

where p, q, g are continuous on (α, β) and $t_0 \in (\alpha, \beta)$. Then, there is exactly one solution $y(t)$ of this problem, and the solution is defined on the interval (α, β).

- The IVP has a solution!
- The solution is unique!
- The unique solution is defined (at least) on the interval (α, β), where is at least twice differentiable

Homogeneous equations with constant coefficients

Definition

To the equation

$$ay'' + by' + cy = 0 \quad a, b, c\mathbb{R}$$

we associate a **characteristic equation**

$$ar^2 + br + c = 0.$$

If the characteristic equation has

1. **two different real roots** $r_1, r_2 \in \mathbb{R} \Rightarrow y = C_1 e^{r_1 t} + C_2 e^{r_2 t},\ C_1, C_2 \in \mathbb{R}$
2. **double root** $r \in \mathbb{R} \Rightarrow y = C_1 e^{rt} + C_2 t e^{rt},\ C_1, C_2 \in \mathbb{R}$
3. **two complex roots** $r = \alpha \pm i\beta \Rightarrow y = C_1 e^{\alpha t} \cos(\beta t) + C_2 e^{\alpha t} \sin(\beta t)\ C_1, C_2 \in \mathbb{R}$

Why?

1st ingredient

If the characteristic equation associated to $ay'' + by' + cy = 0$ has

1. **two different real roots** $r_1, r_2 \in \mathbb{R} \Rightarrow y = C_1 e^{r_1 t} + C_2 e^{r_2 t},\ C_1, C_2 \in \mathbb{R}$
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Theorem

If $\tilde{r}$ is a root of the characteristic equation , then $y(t) = e^{\tilde{r}t}$ is a solution.

Corollary

$ar^2 + bt + c = 0$ has two different roots, r_1 and r_2, then $e^{r_1 t}$ and $e^{r_2 t}$ are both solutions.

Theorem 3.2.2

If y_1 and y_2 are solutions of the differential equation

$$P(t)y'' + Q(t)y' + R(t)y = 0$$

then the linear combination $c_1y_1 + c_2y_2$ is also a solution for any constants c_1, c_2.

Corollary

$ar^2 + bt + c = 0$ has two different roots, r_1 and r_2, then e^{r_1t} and e^{r_2t} are both solutions. Furthermore, $C_1e^{r_1t} + C_2e^{r_2t}$ is also a solution for any $C_1, C_2 \in \mathbb{R}$.

Question

Can we solve any IVP?

In other words, can we solve $ay'' + by' + cy = 0$ for any set of initial conditions $y(t_0) = y_0, y'(t_0) = y_0'$?

If so, the solution will be unique! (Why?) In case 1., we need to be able to find C_1 and C_2 such that

$$\begin{cases} C_1 e^{r_1 t_0} + C_2 e^{r_2 t_0} & = y_0 \\ C_1 r_1 e^{r_1 t_0} + C_2 r_2 e^{r_2 t_0} & = y_0' \end{cases}$$

Can we always do that?

$$\begin{cases} \underbrace{C_1}_{x} \underbrace{e^{r_1 t_0}}_{a} + \underbrace{C_2}_{y} \underbrace{e^{r_2 t_0}}_{b} & = \underbrace{y_0}_{e} \\ \underbrace{C_1}_{x} \underbrace{r_1 e^{r_1 t_0}}_{c} + \underbrace{C_2}_{y} \underbrace{r_2 e^{r_2 t_0}}_{d} & = \underbrace{y_0'}_{f} \end{cases} \Leftrightarrow \begin{cases} ax + by & = e \\ cx + dy & = f \end{cases}$$

Solving the RHS algebraically we get

$$x = \frac{1}{ad - bc}(de - bf)$$
$$y = \frac{1}{ad - bc}(af - ce)$$

For generic e and f we need to be able to divide by $ad - bc$ to solve the system.

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

Determinant

Other way to look at it. Each equation in

$$\begin{cases} ax + by &= e \\ cx + dy &= f \end{cases}$$

represents a line in the *xy*-plane.

Thus, we are trying to find where they intersect.

- If $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$, the lines are **not** parallel and there is only one solution.
- If $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$, the lines are parallel.
  - The lines are parallel and distinct $\rightsquigarrow$ no solution.
  - Both equations represent the same line $\rightsquigarrow$ infinitely many solutions.

Back to solving the IVP

$$\begin{cases} C_1 e^{r_1 t_0} + C_2 e^{r_2 t_0} = y_0 \\ C_1 r_1 e^{r_1 t_0} + C_2 r_2 e^{r_2 t_0} = y_0' \end{cases}$$ has a solution **if and only if** $\det \begin{pmatrix} e^{r_1 t0} & e^{r_2 t_0} \\ r_1 e^{r_1 t_0} & r_2 e^{r_2 t_0} \end{pmatrix} \neq 0$

$$\det \begin{pmatrix} e^{r_1 t0} & e^{r_2 t_0} \\ r_1 e^{r_1 t_0} & r_2 e^{r_2 t_0} \end{pmatrix} = \underbrace{e^{(r_1 + r_2) t_0}}_{\neq 0} \underbrace{(r_1 - r_2)}_{\neq 0} \neq 0$$

Can we generalize this?

Definition

Let y_1 and y_2 be two solutions of the homogeneous ODE

$$y'' + p(t)y' + q(t)y = 0.$$

The **Wronkskian** of these solutions is a function in t

$$W(y_1, y_2)(t) := \det \begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix} = y_1(t)y_2'(t) - y_2(t)y_1'(t)$$

Exercise 20 (practice)

$W(f, g)(t) = t\cos t - \sin t$;
$u = f + 3g$ and $v = f - g$.

Find $W(u, v)(t)$.

Theorem 3.2.3

Let $y_1(t)$ and $y_2(t)$ be two solutions of the homogeneous ODE

$$y'' + p(t)y' + q(t)y = 0,$$

that are defined at t_0. If $W(y_1, y_2)(t_0) \neq 0$, then **every** solution to the initial value problem

$$y(t_0) = y_0, \quad y'(t_0) = y'_0$$

can solved in the form

$$y(t) = C_1 y_1(t) + C_2 y_2(t).$$