Math 23, Spring 2017

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March 31st, 2017

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Quick recap

Last time:

- We learned some vocabulary, ordinary/partial, linear/nonlinear, order, etc..
- We learned how to solve $y' + p(t)y = g(t)$
- The general solutions looks like

$$y(t) = \frac{1}{\mu(t)} \int \mu(t)g(t) \, \mathrm{d}t$$

$$\text{with } \mu(t) = e^{\int p(t) \, \mathrm{d}t}$$

Today:

- Separable equations (we already have seen these ones a couple of times...)
- Existence and uniqueness of solutions

Separable Equations

A general first order ODE can be written in the form

$$M(x, y) + N(x, y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \tag{1}$$

for some functions M, N.

If M is a function of only x and N is a function of only y, then

$$M(x) + N(y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0\,.$$

Then

$$\int N(y)\,\mathrm{d}y = -\int M(x)\,\mathrm{d}x.$$

This is known as **separable** equation, as we have separated the variables x and y.

Separable equations, Example

Exercise 2.2.23

Solve the IVP

$$y' = 2y^2 + xy^2, \quad y(0) = 1$$

and determine where the solution attains minimum value.

- Order? Linear/Nonlinear? Separable?
- $$y(x) = -1/\left(2x + \frac{x^2}{2} - 1\right)$$
- To find the minimum we either start to take derivatives of $y(x)$ or we use the equation that was given to us!
- local minimum at $x = -2$

A classical and important example

Example 2.4.3

Solve the IVP : $y' = y^{1/3}, \quad y(0) = 0$

- Order? Linear/Nonlinear? Separable?

$$y' = y^{1/3} \implies \int y^{-1/3}\, \mathrm{d}y = \int \mathrm{d}x$$

$$\implies \frac{3}{2} y^{2/3} = x + C \implies y(x) = \pm\sqrt{\left(\frac{2}{3}(x + C)\right)^3}$$

- $y(0) = 0 = \pm\sqrt{\left(\frac{2}{3}(0 + C)\right)^3} \Rightarrow C = 0$
- We missed one obvious solution! Which one?

A classical and important example

Example 2.4.3

Solve the IVP : $y' = y^{1/3}, \quad y(0) = 0$

We found three solutions to the IVP

1. $y(x) = 0$
2. $y(x) = \sqrt{\left(\frac{2}{3}x\right)^3}$
3. $y(x) = -\sqrt{\left(\frac{2}{3}x\right)^3}$

Question

Are there more solutions?

A classical and important example

Example 2.4.3

Solve the IVP : $y' = y^{1/3}, \quad y(0) = 0$

For $x_0 > 0$ put

$$y^{\pm}_{x_0}(x) = \begin{cases} 0 & x \leq x_0 \\ \pm\sqrt{\left(\frac{2}{3}(x - x0)\right)^3} & x > x_0 \end{cases}$$

Claim

All the functions $y^{\pm}_{x_0}(x)$ are solutions to the IVP.

Theorem 2.4.1

Suppose the functions p, g are continuous on the interval (α, β) containing the point t_0. Then there exists a unique function $y(t)$ that satisfies the DE

$$y' + p(t)y = g(t)$$

for each t in (α, β) and also satisfies the initial condition $y(t_0) = y_0$, where y_0 is an arbitrary initial value.

Exercise 2.4.5

Determine the interval on which the solution of the IVP is **certain** to exist.

$$(4 - t^2)y' + 2ty = 3t^2, \quad y(1) = 3$$

Existence and Uniqueness – non-linear ODEs

Theorem 2.4.2

Suppose the functions $f(t, y)$ and $\dfrac{\partial f}{\partial y}(y, t)$ are continuous on some rectangle $(\alpha, \beta) \times (\gamma, \delta)$ containing the point (t_0, y_0).

Then, in some interval $(t_0 - h, t_0 + h)$ contained in (α, β), there is a unique solution of the IVP

$$y' = f(t, y), \qquad y(t_0) = y_0 \,.$$

Exercise 2.4.15

Solve the IVP

$$y' + y^3 = 0, \quad y(0) = y_0$$

and determine how the interval where the solution exists depend on y_0.

Solution Exercise 2.4.15

- Linear or nonlinear?
- Theorem 2.4.1 does not apply here. Why?
- Theorem 2.4.2 does not say anything about the domain of definition of the solution (right?)
- However, Theorem 2.4.2 tells us that the solution will be unique around $t_0 = 0$
- Rewrite the DE as $\dfrac{-y'}{y^3} = 1$, and note that we lost a solution
- Thus $y(x) = \begin{cases} \pm\sqrt{\frac{1}{2(x+C)}}, & y_0 \neq 0 \\ 0, & y_0 = 0 \end{cases}$
- if $y_0 \neq 0$, then $y(0) = y_0 = \pm\sqrt{\dfrac{1}{2(0+C)}} \Rightarrow y_0^2 = \dfrac{1}{2C} \Rightarrow C = \dfrac{1}{2y_0^2}$

Solution Exercise 2.4.15 Continued

- $$y(x) = \begin{cases} \sqrt{\frac{1}{2\left(x+\frac{1}{2y_0^2}\right)}}, & y_0 > 0 \\ 0, & y_0 = 0 \\ -\sqrt{\frac{1}{2\left(x+\frac{1}{2y_0^2}\right)}}, & y_0 < 0 \end{cases}$$
- If $y_0 = 0$, $y(x) = 0$ is defined in $\mathbb{R}$
- In the other two cases we need to ask for what x we have $2\left(x + \frac{1}{2y_0^2}\right) > 0$, which is
$$x \in [-\frac{1}{2y_0^2}, +\infty)$$
- Answer: if $y_0 \neq 0$ the solution is defined on $[-\frac{1}{2y_0^2}, +\infty)$, for $y_0 = 0$, the solution is defined in $\mathbb{R}$