Math 23, Spring 2017

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Solving homogeneous linear systems

$$X' = A \cdot X$$

$X(t) = e^{\lambda t} v$ a solution $\Leftrightarrow$ v is an eigenvector and λ is its corresponding eigenvalue.

3 possible cases:

(A) All eigenvalues are real and distinct. ✓

(B) Some come in complex conjugate pairs ✓

(C) Some eigenvalues come with multiplicity greater than 1.

Very similar to solving $ay'' + by' + cy = 0$.

Repeated eigenvalues

Example 0

Solve $X' = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} X$

1. Try using eigenvectors and eigenvalues.
2. Try solving one equation at a time.

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Fact

If λ is a root of $\det(A - \lambda I)$ with multiplicity $j > 0$.

The vector space of eigenvectors corresponding to λ can have dimension $< j$, i.e., we might not be able to find j independent eigenvectors.

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Solution: $X(t) = c_1 e^{\lambda t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{\lambda t} \left(t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$

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- $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector $\Leftrightarrow Av = \lambda v$
- $w = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is a generalized eigenvector $\Leftrightarrow Aw = v + \lambda w$

Generalized eigenvectors (aka Secondary eigenvectors)

Definition

Given an eigenvector corresponding to λ ($Av = \lambda v$), the generalized eigenvector w associated to v satisfies

$$(A - \lambda I)w = v \Leftrightarrow Aw = v + \lambda w$$

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$$\begin{aligned}
\phi(t) &= e^{\lambda t} w + t e^{\lambda t} v \\
A\phi(t) &= e^{\lambda t}\left(Aw + tAv\right) \\
&= e^{\lambda t}(\lambda w + v + \lambda t v) \\
&= \phi'(t)
\end{aligned}$$

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Note

Generalized eigenvectors appear only if the number of independent eigenvectors associated to λ (dimension of the eigenspace) is **less** multiplicity of the eigenvalue.

Exercise 7.8.1

Solve $X' = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} X$

Recall:

- eigenvector $(A - \lambda I)v = 0 \rightsquigarrow e^{\lambda t} v$ is a solution
- eigenvalue $(A - \lambda I)w = v \rightsquigarrow e^{\lambda t} w + t e^{\lambda t} v$ is a solution

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$v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $w = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Generalized eigenvectors

The process can keep going on if you don't have enough solutions

- $(A-\lambda I)v = 0 \rightsquigarrow e^{\lambda t}v$
- $(A-\lambda I)w_1 = v \rightsquigarrow e^{\lambda t}w_1 + te^{\lambda t}v$
- $(A-\lambda I)w_2 = w_1 \rightsquigarrow e^{\lambda t}w_2 + te^{\lambda t}w_1 + \frac{t^2}{2}e^{\lambda t}v$
- ...
- $(A-\lambda I)w_k = w_{k-1} \rightsquigarrow e^{\lambda t}w_k + te^{\lambda t}w_{k-1} + \cdots + \frac{t^k}{k!}e^{\lambda t}v$

Jordan blocks

$$X' = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix} \cdot X$$

$$\Rightarrow X(t) = c_1 e^{\lambda t} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + c_2 e^{\lambda t} \begin{pmatrix} t \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + c_3 e^{\lambda t} \begin{pmatrix} t^2/2 \\ t \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \cdots + c_n e^{\lambda t} \begin{pmatrix} t^n/n! \\ t^{n-1}/(n-1)! \\ t^{n-2}/(n-2)! \\ \vdots \\ t \\ 1 \end{pmatrix}$$

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We can always "transform" a matrix so that the only thing that shows up are blocks like the above. For more, search for **Jordan normal form**.

Generic example with block matrices

Solve

$$X' = \begin{pmatrix} \lambda_1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_4 \end{pmatrix} \cdot X$$

Exercise 7.8.18 (homework)

Solve $X' = A \cdot X$, with $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{pmatrix}$ and given that $\lambda = 2$ is a triple root of the characteristic polynomial of A.