Math 23, Spring 2017

Edgar Costa

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Dartmouth College

$$X' = A \cdot X$$

$X(t) = e^{\lambda t} v$ a solution $\Leftrightarrow$ v is an eigenvector and λ is its corresponding eigenvalue.

3 possible cases:

(A) All eigenvalues are real and distinct. ✓

(B) Some come in complex conjugate pairs

(C) Some eigenvalues come with multiplicity greater than 1.

Very similar to solving $ay'' + by' + cy = 0$.

Complex eigenvalues

Claim

Assume that A is an $n \times n$ real matrix. If $\alpha + \beta i$ is an eigenvalue and v is an eigenvector corresponding to $\alpha + i\beta$, then $\alpha - i\beta$ is an eigenvector and $\overline{v}$ (complex conjugate entrywise) is an eigenvector corresponding to $\alpha - i\beta$.

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Ideas:

- $\overline{A} = A$
- $\det(A - \lambda I)$ is a polynomial with real coefficients
- $\overline{Av} = A\overline{v}$

Real solutions

Write $\lambda = \alpha + i\beta$ and $v = a + ib$, such that $Av = \lambda v$.

$$\begin{cases} X_1(t) = e^{\lambda t} v & = e^{\alpha t}(\cos(\beta t) + i \sin(\beta t))(a + ib) \\ X_2(t) = \overline{X_1(t)} = e^{\overline{\lambda} t}\overline{v} & = e^{\alpha t}(\cos(\beta t) - i \sin(\beta t))(a - ib) \end{cases}$$

two **complex** solutions for

$$X' = A \cdot X$$

How can I combine them to get two **real** solutions?

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$$\begin{cases} \tilde{X}_1 = \frac{X_1 + X_2}{2} = e^{\alpha t}(\cos(\beta t)a - \sin(\beta t)b) & (= \operatorname{Re}(X_1)) \\ \tilde{X}_2 = \frac{X_1 - X_2}{2i} = e^{\alpha t}(\sin(\beta t)a + \cos(\beta t)b) & (= \operatorname{Im}(X_1)) \end{cases}$$

One can show that they are linearly independent.

Exercise 7.6.1

Solve $X' = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} \cdot X$

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$$\lambda = 1 \pm 2i$$

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The eigenvector corresponding to $1 + 2i$ is $v = \begin{pmatrix} 1 \\ 1 - i \end{pmatrix}$

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$$v_{-2} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \text{ and } v_{-1+i\sqrt{2}} = \begin{pmatrix} \frac{\sqrt{2}}{i+\sqrt{2}} \\ -\frac{i}{i+\sqrt{2}} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ 1 \end{pmatrix} + i \begin{pmatrix} -\frac{\sqrt{2}}{3} \\ -\frac{\sqrt{2}}{3} \\ 0 \end{pmatrix}$$